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MAXIMUM LIKELIHOOD ESTIMATION OF SEISMIC EVENT MAGNITUDE FROM NETWORK DATA

Frode Ringdal

Texas Instruments, Incorporated

Prepared for:

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MAXIMUM LIKELIHOOD ESTIMATION OF SEISMIC EVENT MAGNITUDE FROM NETWORK DATA

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VELA NETWORK EVALUATION AND AUTOMATIC PROCESSING RESEARCH

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By assuming a normal distribution of world-wide magnitudes for any given event, it is possible to establish a simple statistical model that

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In this report, maximum likelihood estimation is applied to determine event magnitude based on this model. The advantages and limitations of the technique are discussed using both simulated and real data. It is found that the maximum likelihood method, when applied properly, has the potential to yield a significant improvement in network magnitude estimates compared to the conventional averaging technique.

ABSTRACT

Seismic networks often tend to overestimate the magnitude of earthquakes, because those stations within the network that do not detect a particular event are ignored in the conventional magnitude averaging procedure.

By assuming a normal distribution of world-wide magnitudes for any given event, it is possible to establish a simple statistical model that includes the additional information that the event magnitude at non-detecting stations must be below a certain threshold value.

In this report, maximum likelihood estimation is applied to determine event magnitude based on this model. The advantages and limitations of the technique are discussed using both simulated and real data. It is found that the maximum likelihood method, when applied properly, has the potential to yield a significant improvement in network magnitude estimates compared to the conventional averaging technique.

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SECTION I
INTRODUCTION

When estimating the magnitude of an earthquake recorded by a seismic network, the common approach is to average all magnitudes measured at those individual stations that actually detect the event. This procedure often leads to overestimating the magnitudes of events that are near the network detection threshold since many stations will not detect such events, and therefore be ignored in the averaging procedure. Clearly, those stations will usually be the ones with the weakest signal, and the net effect is to introduce a positive bias in the estimation procedure.

Herrin and Tucker (1972) computed the expected error introduced by the above magnitude estimation method for the case of a homogeneous or near-homogeneous network. Their basic assumption was that world-wide bodywave magnitudes of a given event follow a Gaussian distribution with unknown mean μ and variance σ^2. They called μ the 'true' magnitude of the event, and computed the bias relative to this (unknown) value as a function of σ^2 and the network characteristics.

This report uses the same basic hypothesis as Herrin and Tucker regarding the normality of worldwide event magnitudes. However, the estimation problem is approached in a different way. Specifically, we assume that, for a given event, the magnitudes at all detecting stations are measured as well as upper limits on the magnitudes at non-detecting stations. These upper limits are obtained by measuring the largest noise peak within the expected signal arrival window, and computing the corresponding 'noise magnitude'. Then, informally, we find the 'most likely' magnitude based upon this combined information. More precisely, we apply the statistical method of maximum likelihood estimation to the above problem.

Some comments are appropriate regarding the Gaussian model for world-wide event magnitude distribution. Freedman (1967) studied the amplitude distribution of bodywave recordings published by the United States Coast and Geodetic Survey (USCGS), and found a close correspondence with the log normal distribution. Clearly, a lognormal distribution of signal amplitudes implies a normal distribution of magnitudes. This result has been supported by other observational studies. The normality of surface-wave magnitudes has not, to our knowledge, been definitely established, but still seems to be a reasonable assumption.

It is natural to ask whether this statistical picture of world-wide magnitude distribution is consistent with recent advances in earthquake source theory, or, phrased in another way, if most of the observed variance could be predicted from source and regional information. Clearly, the following factors influence the station magnitudes for a given event, and, if known, could be used to compute event parameters in a deterministic way from a limited set of observations:

- Source type and complexity of source time function
- Near-source earth structure
- Attenuation of signal due to transmission path effects
- Near-receiver crustal effects such as scattering of P-waves.

While path and receiver effects may be accurately predicted if a large number of events from a given region are available for calibration purposes, the source and near-source effects on the amplitude patterns of P-waves (in particular) are very difficult to specify. For example, one would expect an underground explosion to produce less variation in world-wide amplitudes than an earthquake; still, Lambert et al. (1970) found the Longshot explosion to exhibit greater amplitude variation than the average earthquake.

It is also difficult to infer the radiation pattern of an earthquake from that of a previous event in the same area. For example, Ringdal (1974) found significant scattering in the magnitude residuals between the LASA and NORSAR arrays for a large aftershock sequence from the Kurile Islands in June, 1973.

Finally, many authors now believe that most large earthquakes are composed by a number of events, possibly with different fault planes (Wyss and Brune, 1967; Blandford, 1974; Burdick and Helmberger, 1974). This viewpoint implies that adequate source solutions may be extremely difficult to find. Clearly, difficulties also arise when processing smaller earthquakes, which may be simpler in nature, but lack available recordings of high signal-to-noise ratio.

In summary, the statistical approach to describing the worldwide event magnitude distribution has the advantage of getting around the considerable difficulties inherent in computing deterministic event radiation patterns, while still retaining a realistic picture of actually observed data. The true magnitude of an event is a parameter of the assumed statistical distribution, and can therefore be estimated from statistical considerations alone.

Section II of this report establishes the maximum likelihood approach to the magnitude estimation problem. In Section III, simulated event data are introduced in order to evaluate the estimator. Section IV applies the method to real seismic data recorded by the World-Wide Standard Seismograph Network (WWSSN) and the Very Long Period Experiment (VLPE) network. Finally, conclusions and recommendations from this study are presented in Section V.

SECTION II

DESCRIPTION OF THE MAXIMUM LIKELIHOOD METHOD

This section gives a brief description of the statistical model leading to the development of the maximum likelihood network magnitude estimator. The likelihood function is established for two separate cases, corresponding to whether the individual station detection thresholds are known by actual measurement or only by statistical distribution. The asymptotic properties of the estimator are also derived.

A. THE GAUSSIAN MODEL

Our basic assumption is that, for a given event, world-wide magnitudes follow a Gaussian distribution with parameters (μ, σ^2). The (unknown) mean of this distribution, μ, is called the true magnitude of the event, and our objective is to estimate this parameter based upon a set of observations.

Further, we assume that at a given station, an event is detected if the station magnitude m exceeds a certain threshold value a. This threshold value may be treated as a random variable, if desired, but we will first assume that a is actually measured as the 'noise magnitude' at the expected time of signal arrival.

Thus, we assume that the probability of detecting a given event may be written as:

$$P(\text{Detect}/\mu, \sigma) = P(m \geq a/\mu, \sigma) = \Phi\left(\frac{\mu - a}{\sigma}\right) \qquad \text{(II-1)}$$

where Φ is the standard cumulative normal distribution function.

More generally, if the threshold magnitude m_T is considered a normally distributed random variable with mean a and variance σ_T^2, and if m and m_T are considered independent, we have:

$$P(\text{Detect}/\mu, \sigma) = P(m \geq m_T/\mu, \sigma) =$$

$$= P(m - m_T \geq 0/\mu, \sigma) . \qquad \text{(II-2)}$$

Since $m-m_T$ is a Gaussian variable, we obtain

$$P(\text{Detect}/\mu, \sigma) = \Phi\left(\frac{\mu - a}{\sigma_1}\right) \qquad \text{(II-3)}$$

where

$$\sigma_1^2 = \sigma^2 + \sigma_T^2 . \qquad \text{(II-4)}$$

This model is illustrated in Figure II-1.

In the following, we will concentrate on the approach with a known (non-random) threshold magnitude, since this parameter can always be measured. Clearly, it is more satisfactory to know the precise threshold magnitude than a statistical distribution, especially when taken into account that the statistical distribution may not always be valid; e.g., in cases of high coda levels following a large earthquake or in the event of instrument malfunctioning.

B. THE LIKELIHOOD FUNCTION

Assume that for a given event, records from a network of n stations are examined. Further, assume that the threshold magnitudes a_i, $i = 1, 2, \ldots n$ are known, and that for those stations that detect the event, a magnitude m_i is computed. Finally, assume that all station observations may be considered independent.

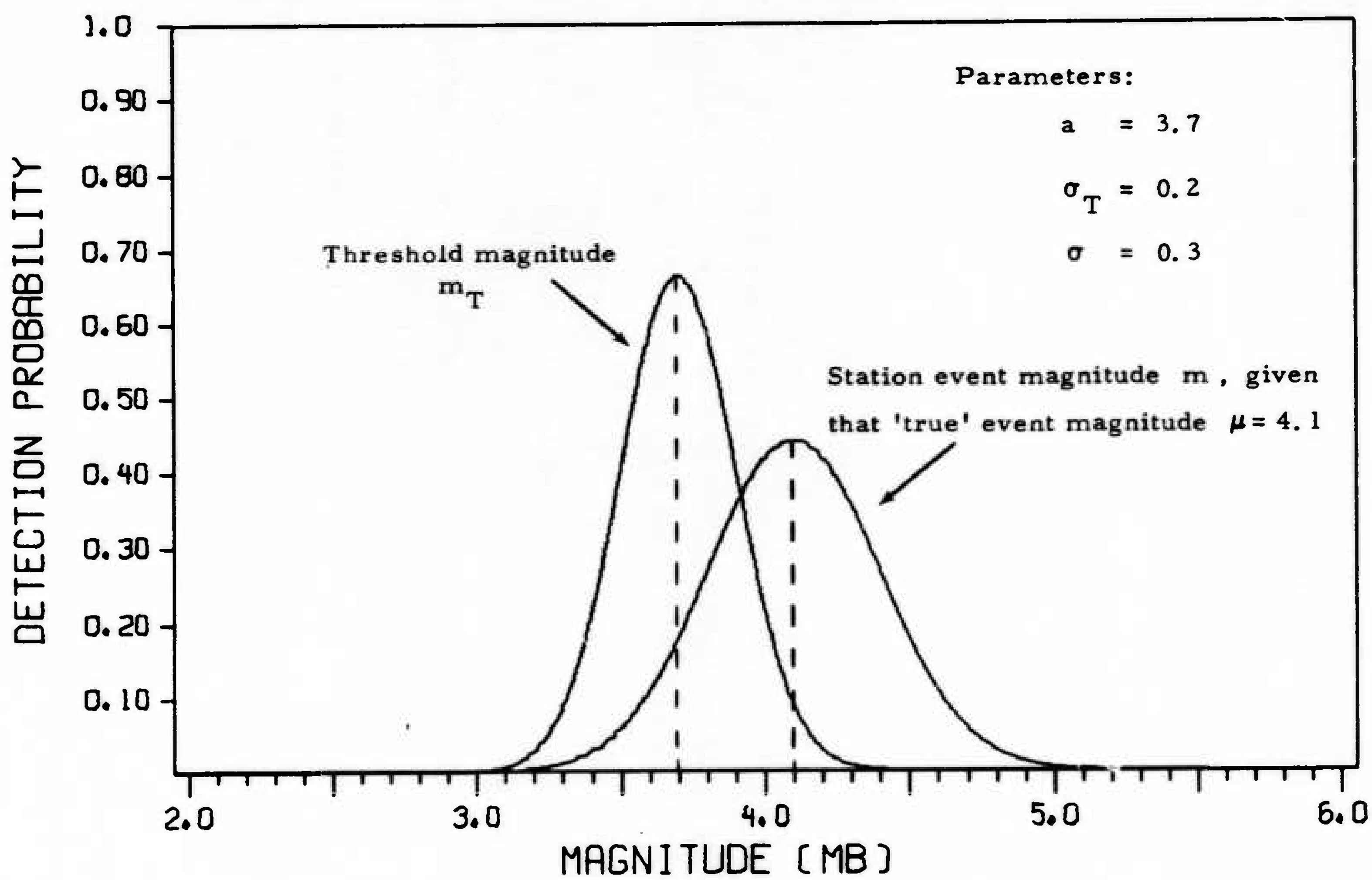


FIGURE II-1

ILLUSTRATION OF THE GAUSSIAN HYPOTHESIS OF STATION DETECTION THRESHOLD MAGNITUDE AND STATION EVENT MAGNITUDE DISTRIBUTION

We want to compute, as a function of the unknown event parameters (μ, σ), the probability that a given situation occurs, and then maximize this likelihood function with respect to the unknown parameters. A formal development of this is presented in Appendix A; the likelihood function is:

$$L(m_1 \ldots m_n / \mu, \sigma) = \prod_{\substack{\text{All} \\ \text{Detections}}} \frac{1}{\sigma} \cdot \phi\left(\frac{m_i - \mu}{\sigma}\right) \cdot \prod_{\substack{\text{All Non-} \\ \text{Detections}}} \Phi\left(\frac{a_i - \mu}{\sigma}\right) \qquad \text{(II-5)}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \qquad \text{(II-6)}$$

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)\, dt \,. \qquad \text{(II-7)}$$

It is of interest to note that the first group of products in (II-5) represents the likelihood function in the special case that all non-detections are ignored, and is maximized by the 'conventional' estimate of μ ; i.e., the average of the magnitudes at the detecting stations.

Since the second factor group in (II-5) is a product of decreasing functions of μ , it follows that our maximum likelihood estimate will always be less than or equal to the conventional estimate, with equality only if all stations detect the event.

Although we are primarily interested in the parameter μ , the likelihood function must in general be maximized as a function of two parameters (μ, σ) . However, it may in some cases be legitimate to restrict σ to assume values within a certain interval, or even to a predefined value. For example, Veith and Clawson (1972) found a value of $\sigma = 0.4$ to be representative of the WWSSN short-period network. Bungum and Husebye (1974) showed by comparing NORSAR and NOAA magnitudes that the standard deviation of 0.3 appeared to be independent of event magnitude. Clearly, the lower standard

deviation for NORSAR than for a WWSSN station is reasonable, since NORSAR magnitudes are averaged over a large number of sensors (more than 100). In conclusion, it appears that the parameter σ may be predicted reasonably well, so that the variation of this parameter may be restricted within a fairly narrow range.

For an inhomogeneous network, it would probably not be right to use the same value of σ for each station. A more general approach allowing individual values of σ as well as individual station bias values is presented in Appendix A.

It is a straightforward extension of the preceding approach to find the likelihood function for the case that the threshold magnitudes m_{Ti} are random variables. Thus, if m_{Ti} is $N(a_i, \sigma^2_{Ti})$ $(i = 1, 2, \ldots n)$:

$$L_1(m_1 \ldots m_n/\mu,\sigma) = \prod_{\substack{\text{All} \\ \text{Detections}}} \frac{1}{\sigma} \cdot \phi\left(\frac{m_i - \mu}{\sigma}\right) \cdot \prod_{\substack{\text{All Non-} \\ \text{Detections}}} \Phi\left(\frac{a_i - \mu}{\sigma_i}\right) \qquad \text{(II-8)}$$

where

$$\sigma_i^2 = \sigma^2 + \sigma_{Ti}^2 . \qquad \text{(II-9)}$$

C. APPROXIMATE CONFIDENCE LIMITS

One of the most prominent features of maximum likelihood estimators is that they often possess very desirable asymptotic properties. Under reasonably general conditions, the following may be proved (Cramer, 1945):

- The solution of the likelihood equation converges in probability to the true parameter value as the number of observations increases
- The maximum likelihood estimator is asymptotically efficient. (Informally, an efficient estimator is one that has a variance lower than any other unbiased estimator)

- The maximum likelihood estimator is asymptotically normally distributed.

In order to verify that these properties apply to our particular case, we find it convenient to regard the selection of network stations as a random experiment, and assume that the probability density function of the threshold magnitude a is of some fixed form $s(a)$. We also assume that individual station selections are independent.

In this way, we can view the estimation procedure as consisting of observing the outcomes (m, a) of n independent experiments, each with the likelihood function:

$$\Lambda(m, a/\mu, \sigma) = \begin{cases} s(a) \cdot \phi\left(\frac{m-\mu}{\sigma}\right) & \text{for } m \geq a \\ s(a) \cdot \Phi\left(\frac{a-\mu}{\sigma}\right) & \text{for } m \leq a \end{cases} \quad \text{(II-10)}$$

It is readily verified that the original likelihood function (II-5) is equivalent to a product of n functions of the form (II-10), in the sense that the factors originating from $s(a)$ do not depend upon μ and σ, and therefore will not influence the maximum likelihood estimates.

As $n \to \infty$, we can now apply the two-dimensional form of the limiting theorem in Cramer (1945), in order to show that the asymptotic properties described above apply to our case.

The important implication of this result is that for a given event, adding new (independent) stations to an already existing network, will cause the maximum likelihood estimate to converge to the true value, even if the new stations added have roughly the same capabilities as the original network stations (i.e., $s(a)$ is kept constant). An example of this principle will be shown in Section III.

It is clearly desirable to obtain an expression for the confidence limits of the maximum likelihood estimator. In view of the preceding considerations, it is reasonable to compute the variance of an unbiased, efficient estimator of μ (usually known as the Cramer-Rao bound) and use this value as an approximation. Such a computation is carried out in Appendix A, under the assumption that σ is known, and the resulting expression is:

$$\operatorname{Var} \hat{\mu} = \sigma^2 \cdot \left[\sum_{i=1}^{n} \left(z_i \, \phi(z_i) + (1 - \Phi(z_i)) + \frac{\left[\phi(z_i)\right]^2}{\Phi(z_i)} \right) \right]^{-1} \tag{II-11}$$

where

$$z_i = \frac{a_i - \mu}{\sigma} \qquad i = 1, 2, \ldots n \tag{II-12}$$

and ϕ and Φ are defined by (II-6) and (II-7).

Examples of using this expression together with the normal distribution to approximate the distribution of the maximum likelihood estimator will be studied in Section III.

SECTION III

SIMULATED PERFORMANCE OF THE ESTIMATOR

A. INTRODUCTION

In general, a closed-form expression of a maximum likelihood estimator may be difficult or impossible to find. Therefore, the exact statistical distribution of the estimator usually cannot be derived (except with respect to asymptotic properties). In many cases, the most practical way to determine the statistical properties of the estimator is to simulate its performance in selected cases. This section presents simulation results for the maximum likelihood network magnitude estimator developed in Section II.

Several different simulation experiments were conducted. Typically, the procedure for each of these was as follows:

- Define a hypothetical seismic network with known threshold magnitudes $a_1, \ldots, a_n$ for each of the n individual stations of the network
- Select an event magnitude μ and standard deviation σ, thereby assuming that the distribution of actual station magnitudes is known
- Simulate 100 events recorded by this network. For each event, n independent, normally distributed random numbers $x_1, x_2, \ldots, x_n$ are generated from the Gaussian distribution (μ, σ^2). These numbers are assigned as station magnitudes for the particular event

- In each of the 100 cases, determine detection/no detection for each of the n stations, by comparing x_i and a_i ($i=1,2,\ldots,n$). Then estimate network event magnitude in the conventional way (by averaging over all detections) and by maximizing the likelihood function (II-5)

- Compare the resulting 100 conventional and maximum likelihood estimates to the expected theoretical distribution of event magnitudes.

In most of our simulation experiments, we chose a 10-station network, with threshold magnitudes in even increments of 0.1 from 4.1 to 5.0 magnitude units. Thus we have for this network, which we call Network 1:

$$a_i = 4.1 + (i-1) * 0.1 \qquad i = 1,2,\ldots 10 \qquad \text{(III-1)}$$

This is thought to be a realistic approach to represent the detection thresholds of a reasonably homogeneous network for a given seismic event. The variations in threshold values would in practical situations be attributed to differences in seismic noise levels, epicentral distances, and signal radiation/propagation effects.

Another network, called Network 2, of 100 stations was also briefly investigated. The threshold levels of this second network were defined as:

$$a_j = 4.1 + (j-1)/10 * 0.1 \qquad j = 1,2,\ldots 100 \qquad \text{(III-2)}$$

where the division sign denotes integer division. Thus, Network 2 had 10 stations of capability 4.1, 10 stations of 4.2 etc. up to 10 stations of 5.0 threshold level.

Event magnitudes μ were selected ranging from 3.5 to 5.5. The standard deviation σ of the world-wide magnitude distribution was set at 0.4 magnitude units in all cases. Finally, we applied the maximum likelihood estimation technique both with σ assumed to be known (Subsection III-B) and with unknown σ (Subsection III-C).

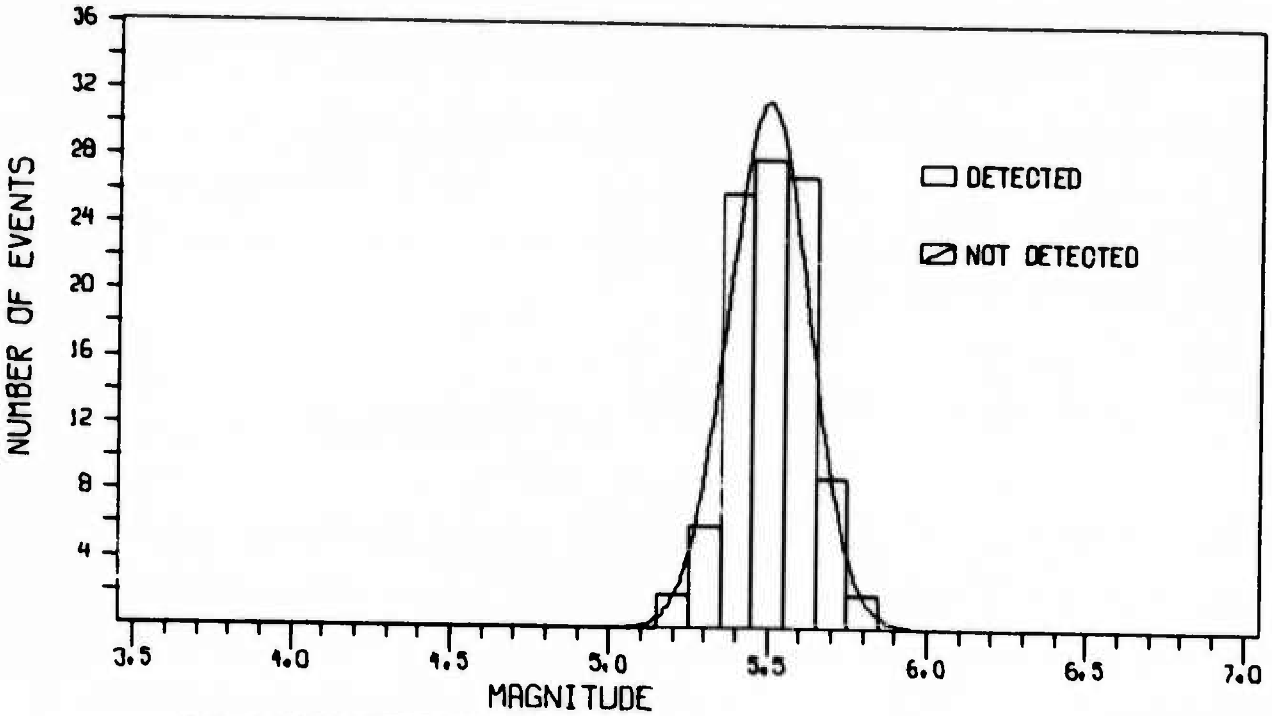
B. SIMULATION WITH KNOWN STANDARD DEVIATION

The first group of simulation experiments was carried out in order to determine the performance of the maximum likelihood estimator under ideal circumstances, i.e., when the correct value of the standard deviation σ of the magnitude distribution was known a priori. Thus, the likelihood function (II-5) in these cases was maximized as a function of one variable μ.

The results are presented in Figures III-1 through III-5, for five different event magnitudes μ, ranging from 5.5 through 3.5. Each figure covers 100 simulated events of identical (true) magnitude, recorded by Network 1 (10 stations). The estimates of μ resulting from applying the conventional method are shown in a histogram at the upper half of each figure, and the corresponding maximum likelihood estimates are shown in the lower half.

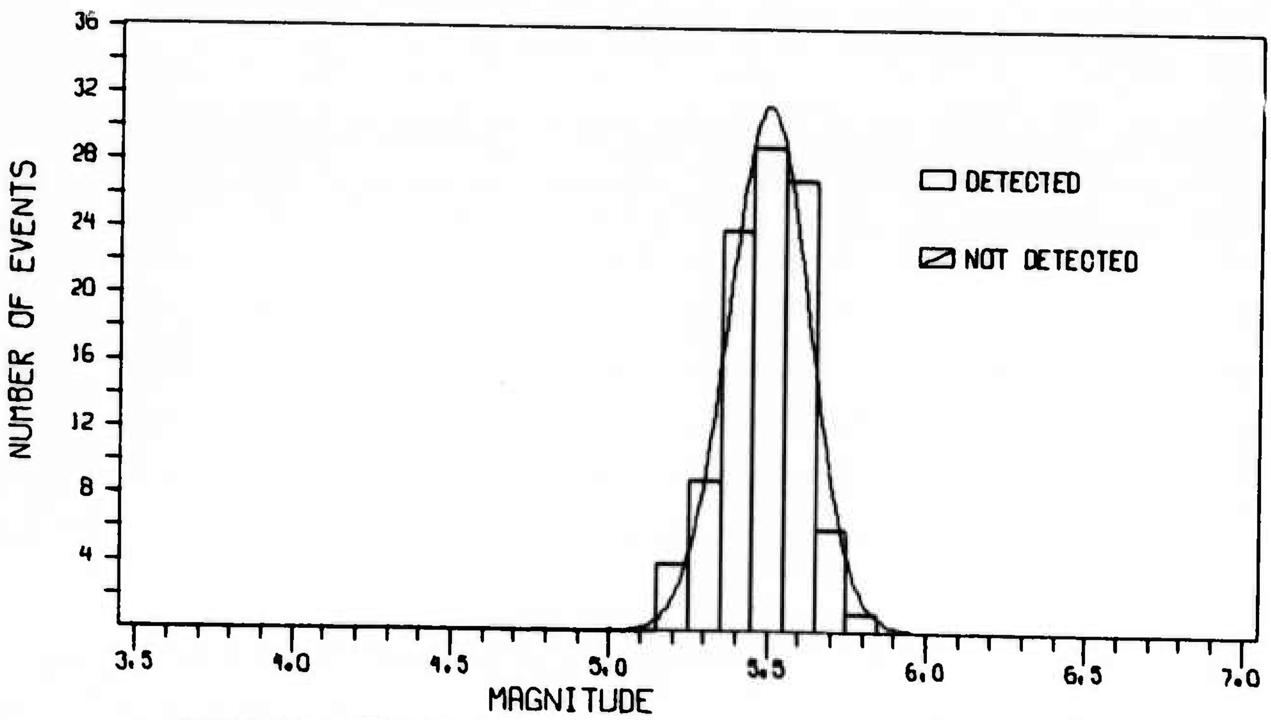
The two smooth curves on each figure are Gaussian distributions centered at the true magnitude. The variance of the upper curve is the theoretical variance for a magnitude estimate by a network of $n = 10$ stations, provided all stations detect. The variance of the lower curve is the Cramer-Rao bound defined by equation (II-11), and this curve therefore gives an indication of how well the maximum likelihood estimator may be approximated by its asymptotic limits.

In some cases an event was not detected by any station within the network. Such events are marked separately on the figures, and of course do not have any associated magnitude.



EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES

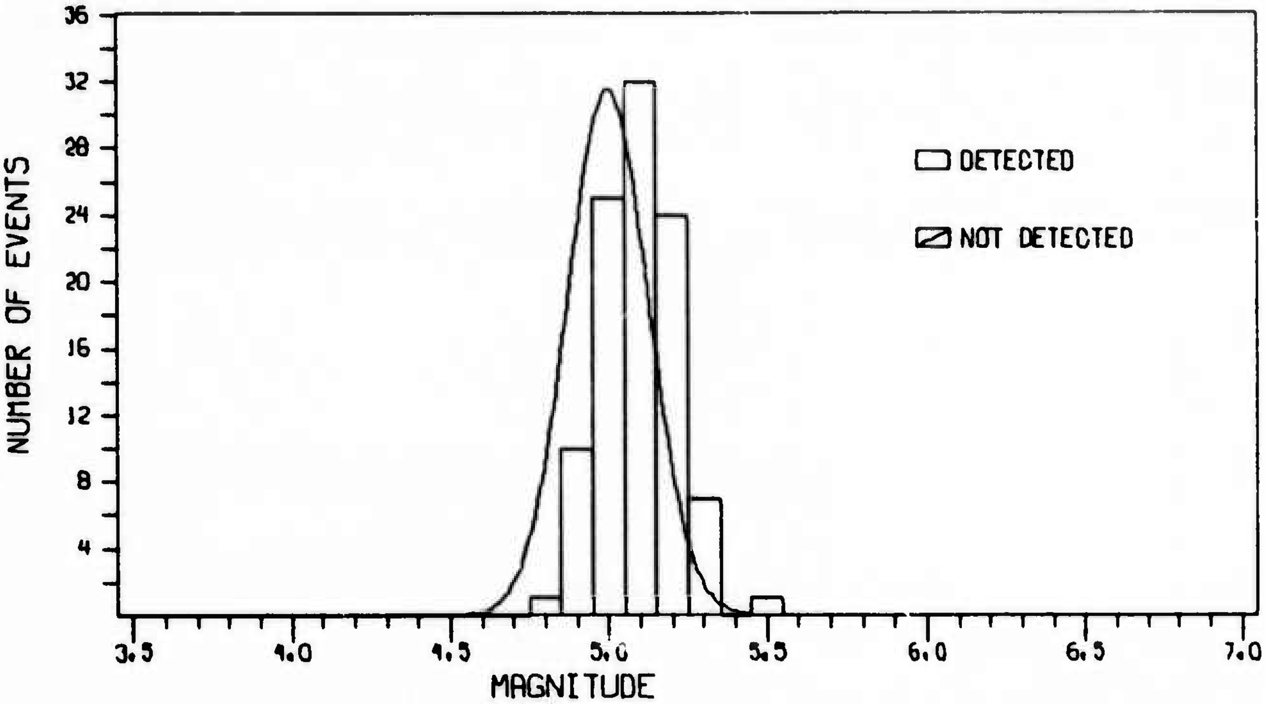
Average Number of Detecting Stations = 9.74 (97.4%)



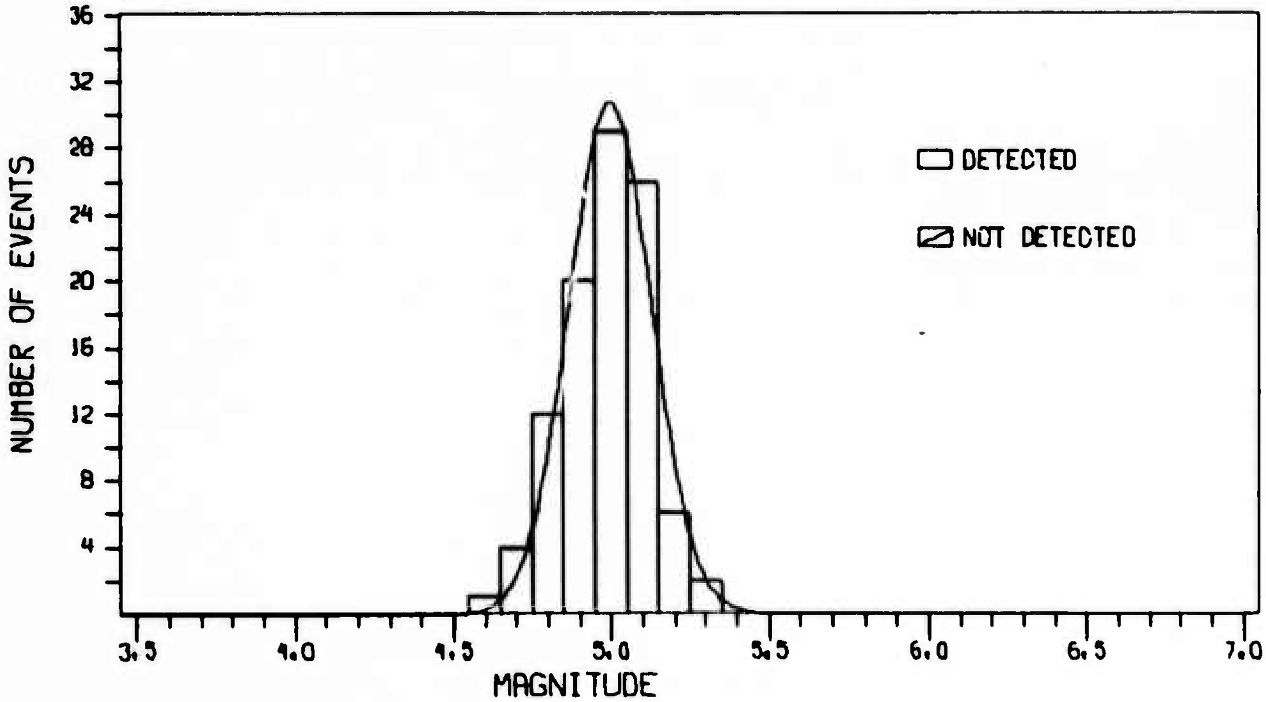
EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (SIGMA=0.40)

FIGURE III-1

SIMULATED PERFORMANCE OF NETWORK 1 FOR 100 EVENTS OF MAGNITUDE 5.5 AND KNOWN STANDARD DEVIATION



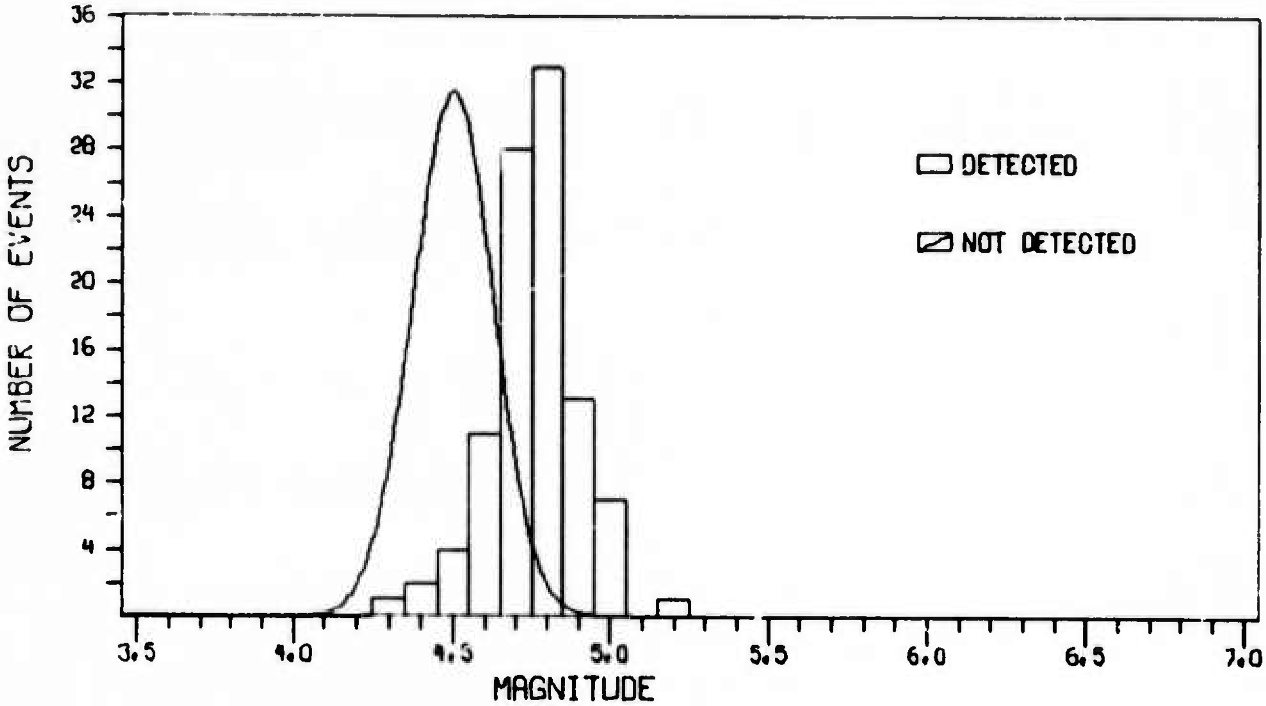
EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES
Average Number of Detecting Stations = 8.14 (81.4%)



EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (SIGMA=0.40)

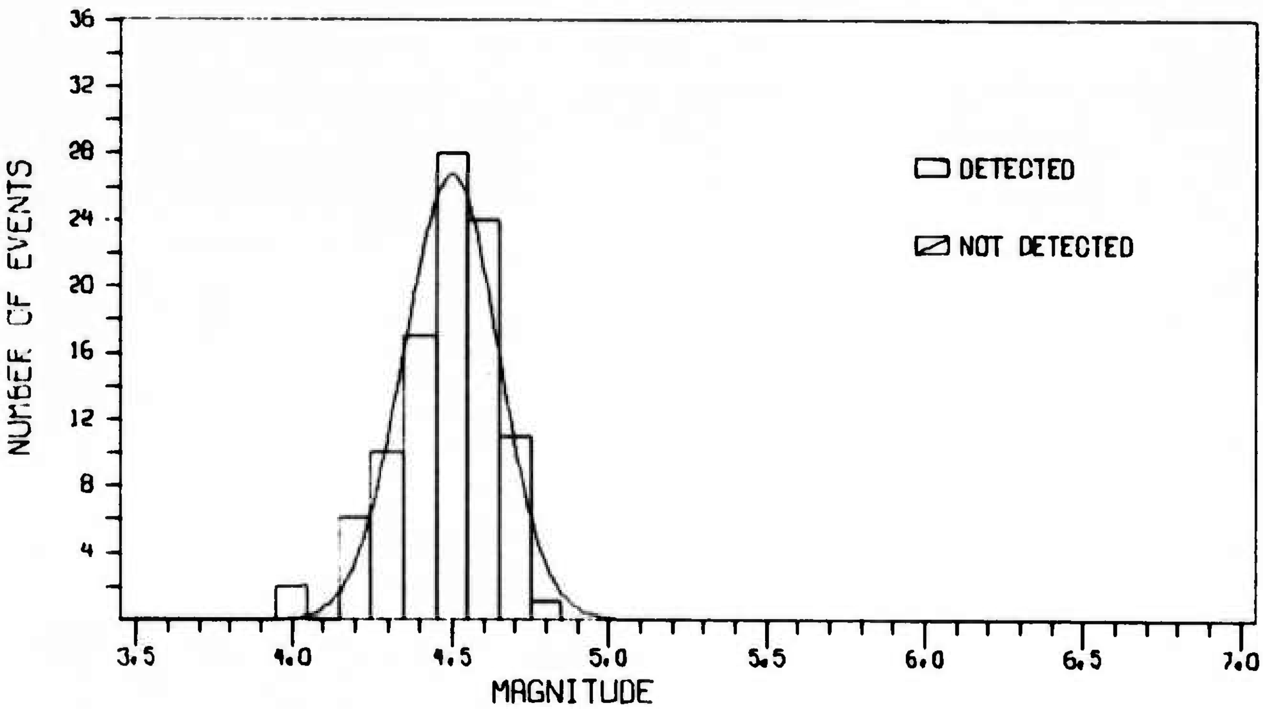
FIGURE III-2

SIMULATED PERFORMANCE OF NETWORK 1 FOR 100 EVENTS OF MAGNITUDE 5.0 AND KNOWN STANDARD DEVIATION



EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES

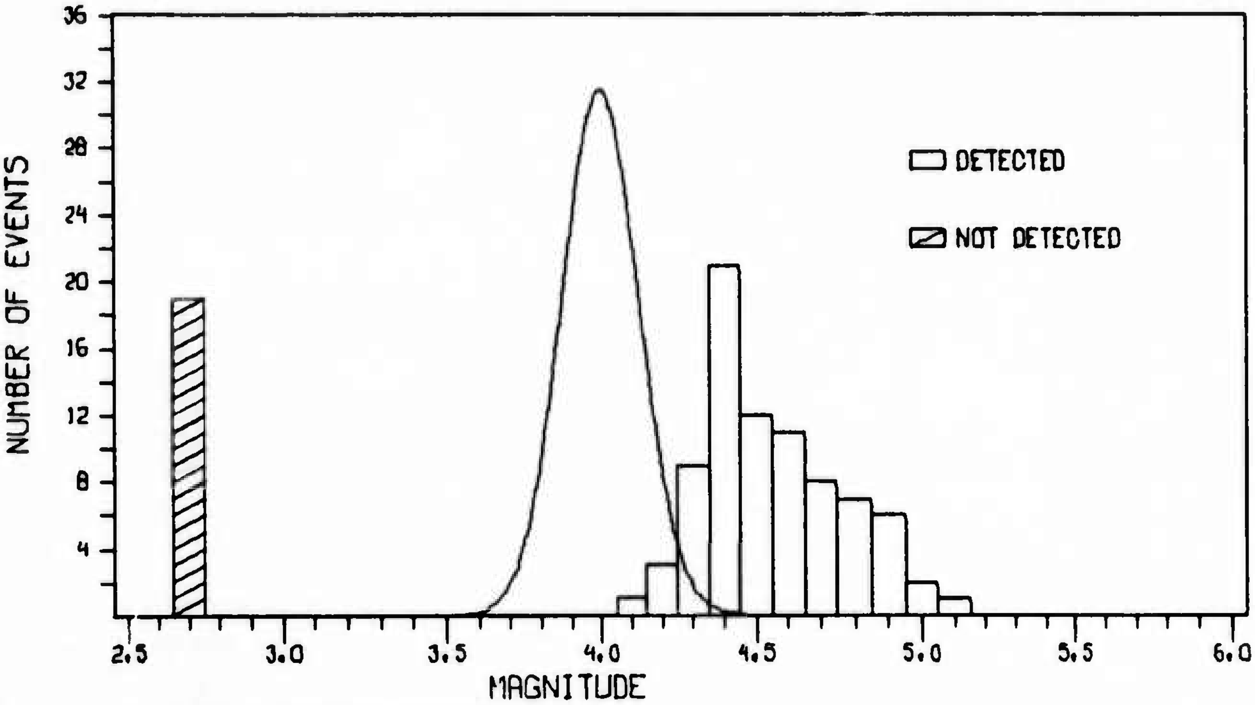
Average Number of Detecting Stations = 4.79 (47.9%)



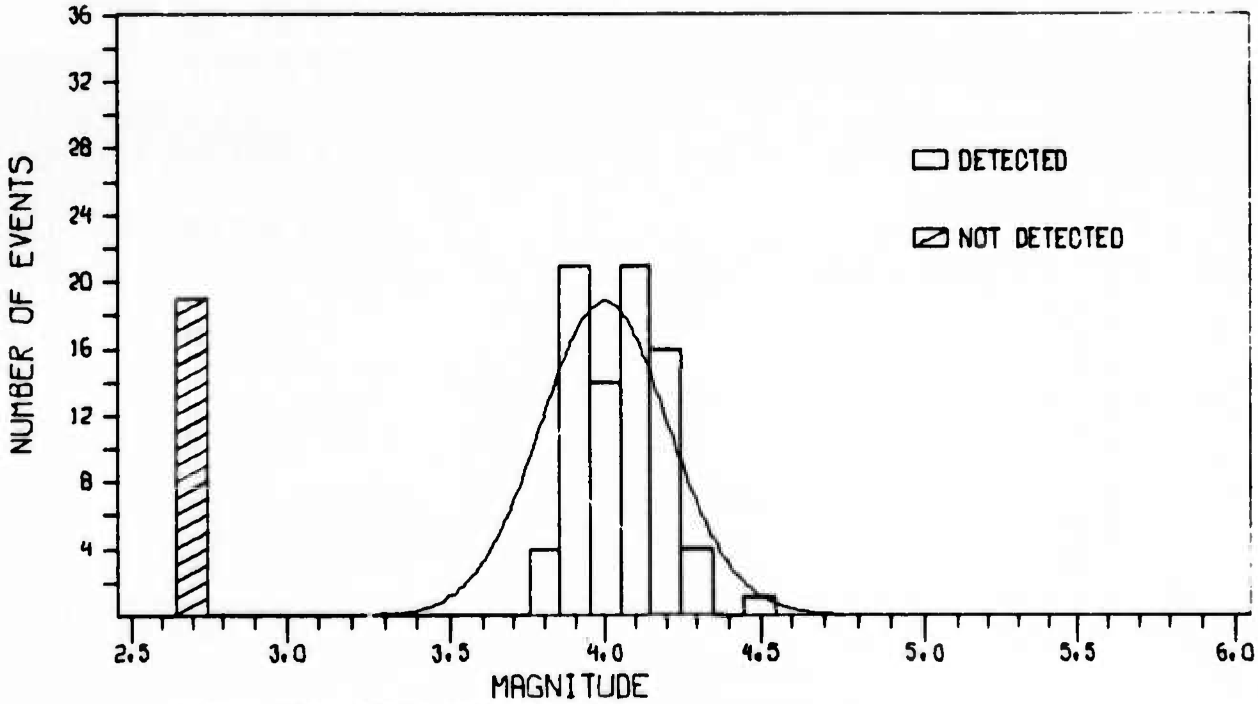
EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (SIGMA=0.40)

FIGURE III-3

SIMULATED PERFORMANCE OF NETWORK 1 FOR 100 EVENTS OF MAGNITUDE 4.5 AND KNOWN STANDARD DEVIATION



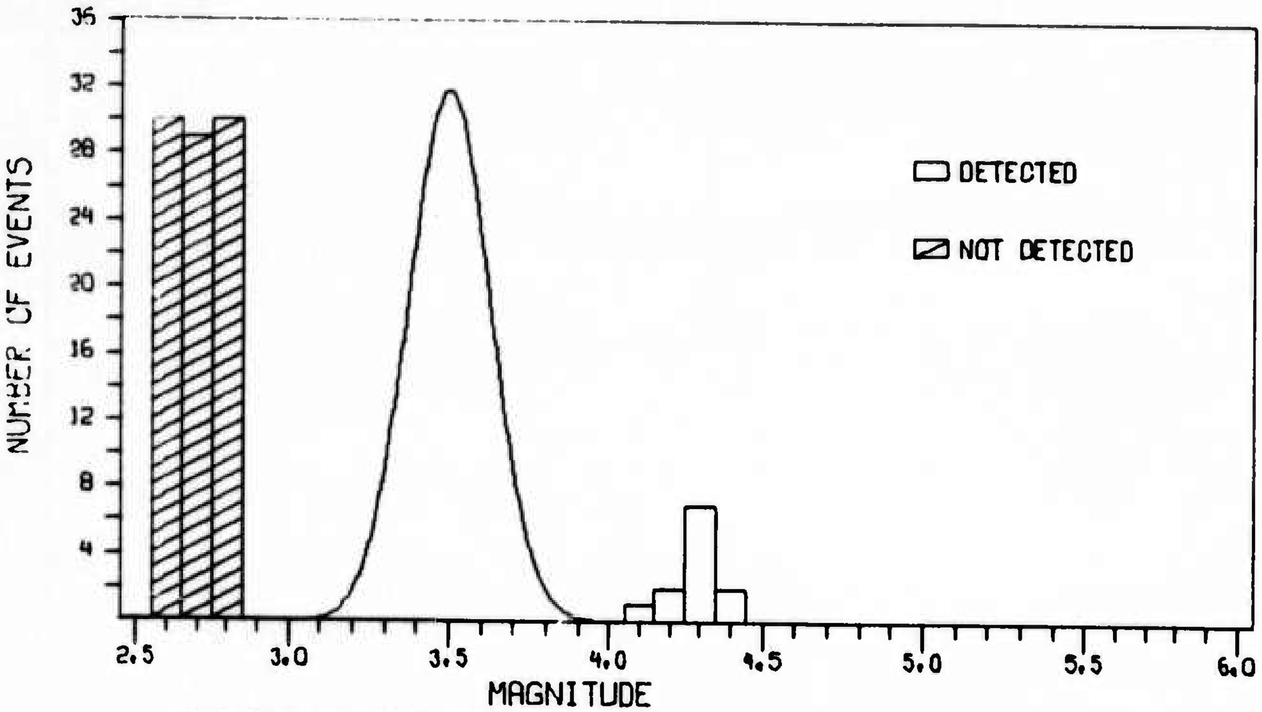
EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES - CASE 3A
Average Number of Detecting Stations = 1.34 (13.4%)



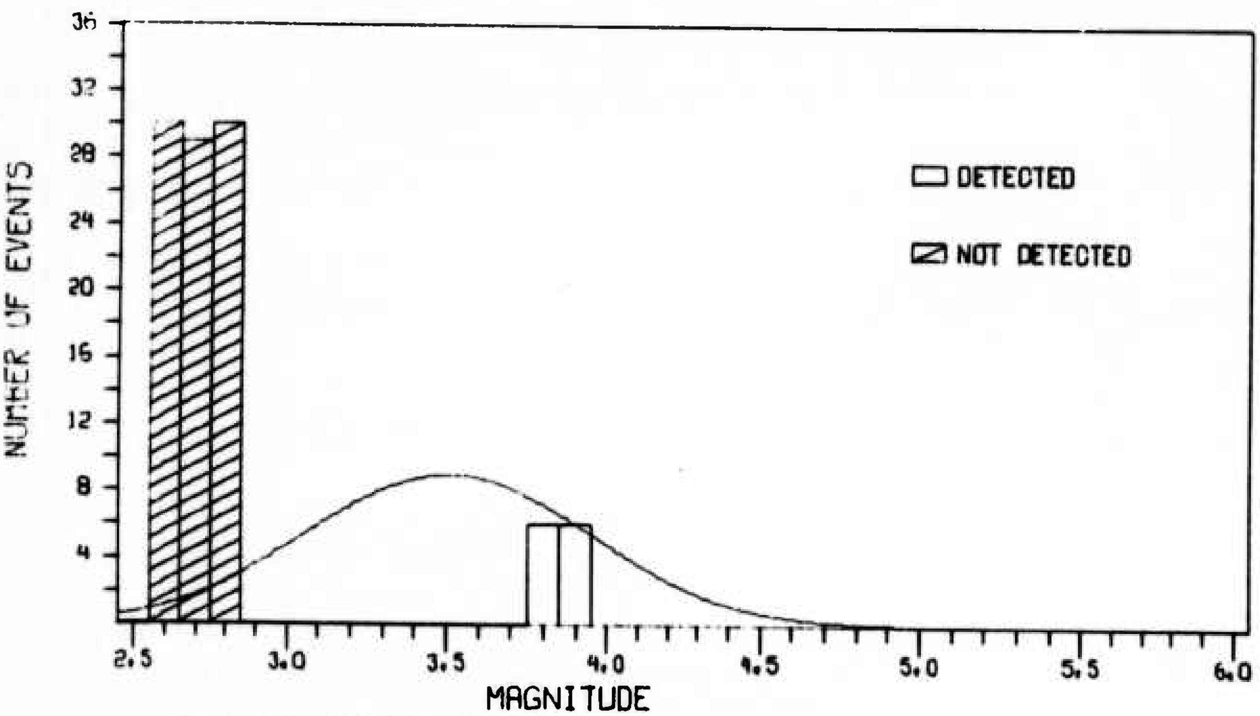
EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (SIGMA=0.40)

FIGURE III-4

SIMULATED PERFORMANCE OF NETWORK 1 FOR 100 EVENTS OF MAGNITUDE 4.0 AND KNOWN STANDARD DEVIATION



EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES
Average Number of Detecting Stations = 0.12 (1.2%)



EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (SIGMA=0.40)

FIGURE III-5

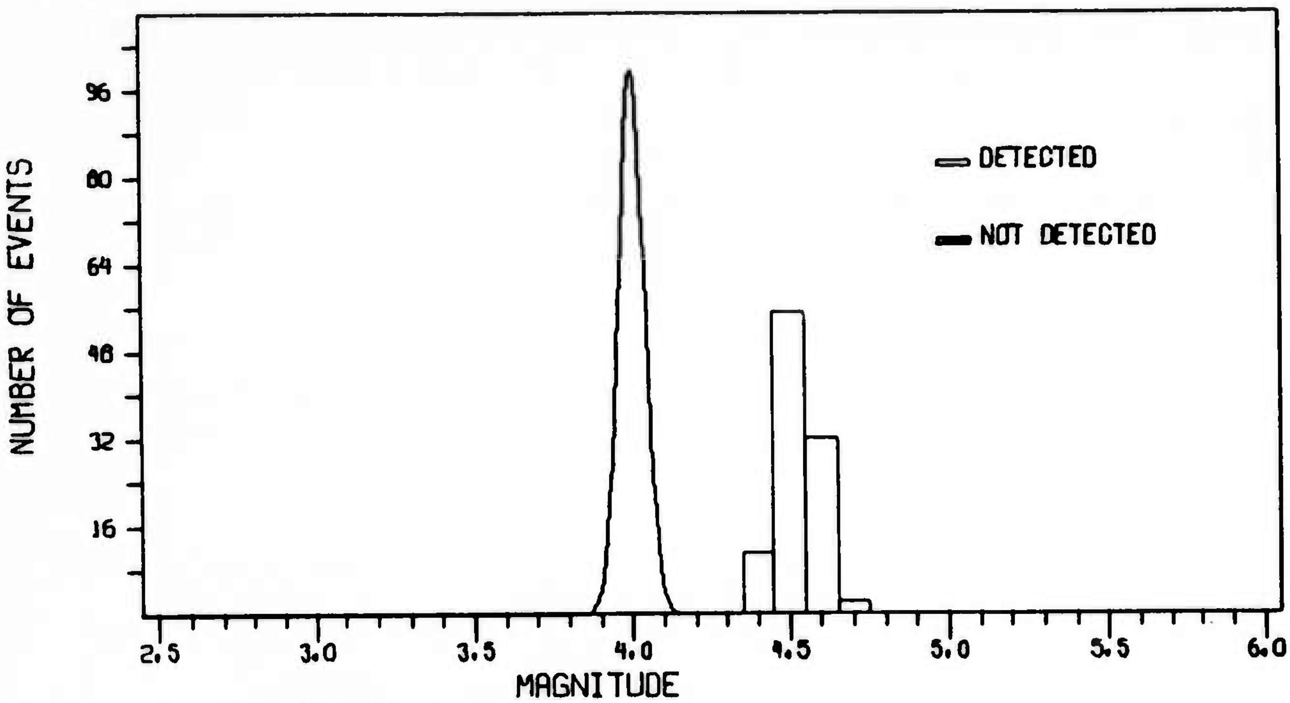
SIMULATED PERFORMANCE OF NETWORK 1 FOR 100 EVENTS OF MAGNITUDE 3.5 AND KNOWN STANDARD DEVIATION

The major results from these simulation experiments are as follows:

- At magnitude 5.5 the two methods are essentially equivalent and unbiased relative to the true magnitude
- At 5.0 and lower magnitudes, the conventional estimates exhibit a significant positive bias. The size of this bias shows a gradual increase from about 0.1 magnitude unit at 5.0 to about 0.5 units at magnitude 4.0
- The maximum likelihood estimates are clearly superior to the conventional estimates, and show essentially zero bias down to magnitude 4.0
- At magnitude 3.5, both methods show less satisfactory performance, mainly due to the low detection probability at this magnitude
- The Cramer-Rao bounds appear to give a good approximation to the actual variance of the maximum likelihood estimator.

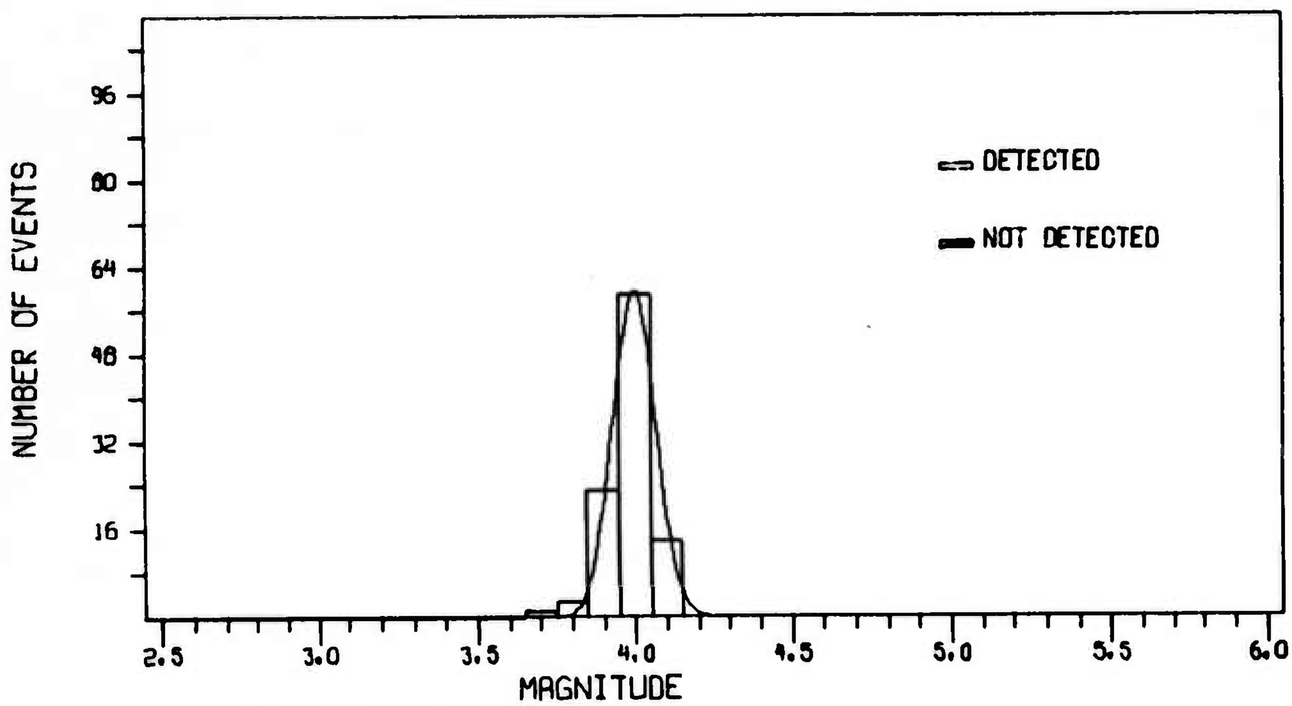
In order to get an impression of what happens when the number of network stations increases, while the average individual station detection capability remains constant, we conducted an experiment with Network 2 (100 stations) as shown in Figures III-6 and III-7 for $\mu = 4.0$ and 3.5, respectively. Clearly, the number of non-detections decreases significantly. More interesting, the following principles are illustrated:

- The maximum likelihood estimates converge to the true magnitude as the number of stations increases
- The conventional estimates converge to a magnitude value that is significantly biased relative to the true value as the number of stations increases.



EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES

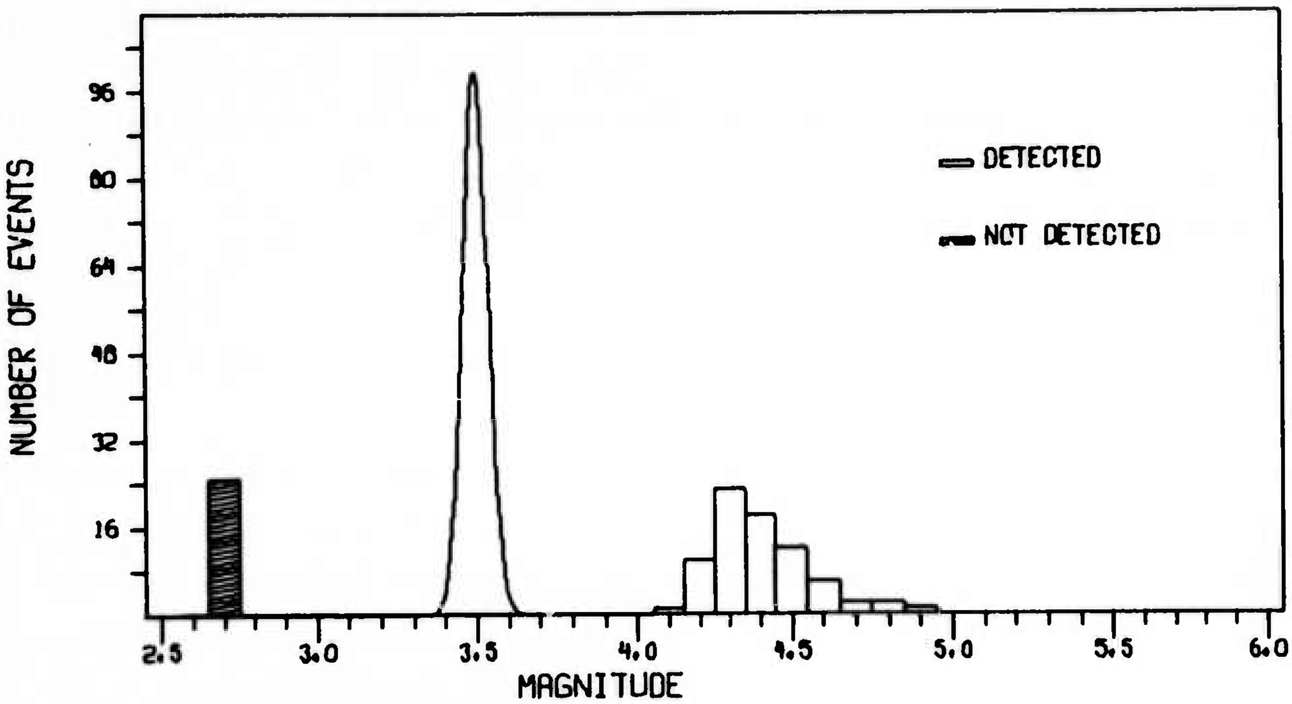
Average Number of Detecting Stations = 12.9 (12.9%)



EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (SIGMA=0.40)

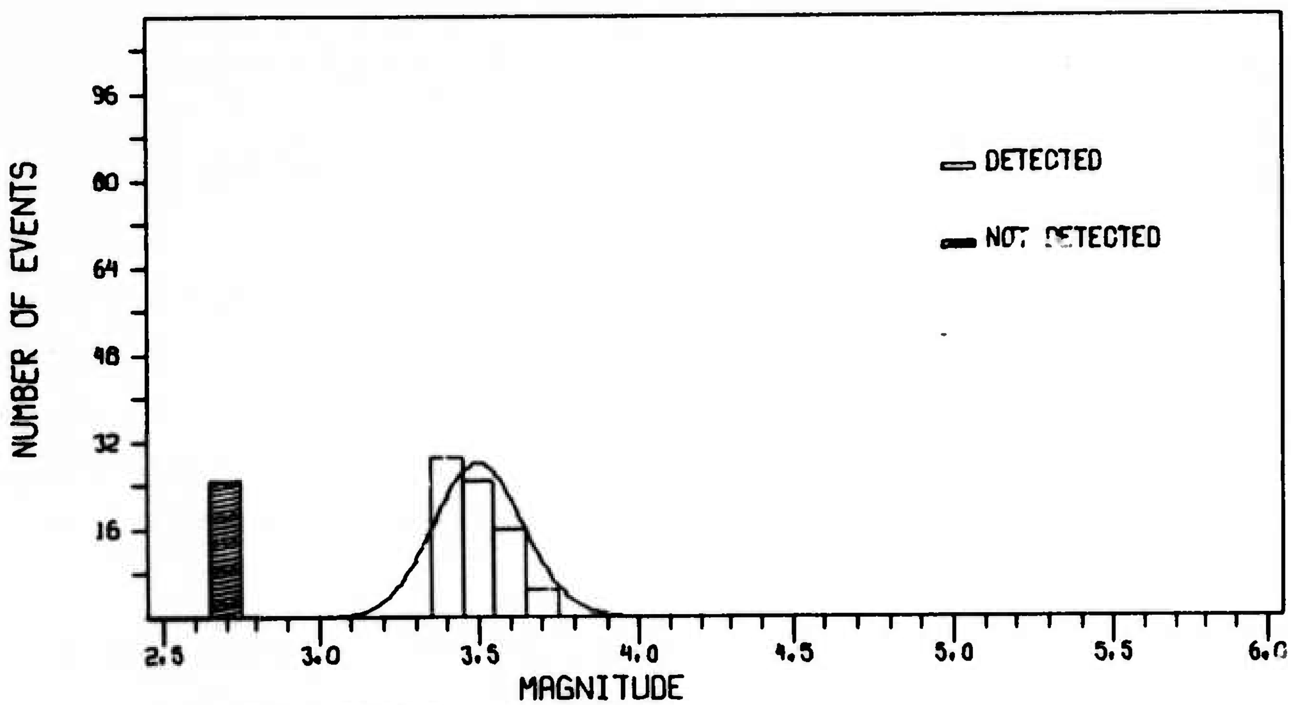
FIGURE III-6

SIMULATED PERFORMANCE OF NETWORK 2 FOR 100 EVENTS OF MAGNITUDE 4.0 AND KNOWN STANDARD DEVIATION



EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES

Average Number of Detecting Stations = 1.27 (1.27%)



EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (SIGMA=0.40)

FIGURE III-7

SIMULATED PERFORMANCE OF NETWORK 2 FOR 100 EVENTS OF MAGNITUDE 3.5 AND KNOWN STANDARD DEVIATION

Thus, if an existing network is augmented with new stations of about the same detection capability, this will improve the maximum likelihood magnitude estimate, but not reduce the bias inherent in the conventional estimate.

A final, interesting observation is that it is possible to use the maximum likelihood method to obtain an upper bound on the network magnitude of an event that is not detected by any individual station. This may be achieved by noting that, for any given situation, there is a certain smallest magnitude that may be estimated by the network. This magnitude corresponds, in the case of either Network 1 or Network 2, to the situation where one station detects the event with magnitude 4.1, and no other station detects. For Network 1, the maximum likelihood estimate in this case is 3.8; for Network 2 it is 3.4.

Such upper bounds on non-detections should of course be used with caution, like all statistical estimates. However, properly interpreted, they may be of value when considering 'negative discriminants' such as the absence of detectable surface-waves for underground explosions in M_s - m_b plots.

C. SIMULATION WITH UNKNOWN STANDARD DEVIATION

The likelihood function (II-5) may be maximized as a joint function of μ and σ provided that the network consists of at least two stations, and that at least one station detected the event. (This last requirement, of course, always applies.) This subsection presents the results of applying two parameter maximization to test situations analogous to those described in Subsection III-B.

It is intuitively clear that any estimate of σ based upon only 10 observations would usually have a fairly large error margin. This is particularly pronounced in those cases when only a few stations detect. In order

to minimize the effects of gross error in the estimate of σ on the resulting estimate of the magnitude μ, we restricted σ to values within a predefined range in the estimation procedures. This range was, somewhat arbitrarily, set to

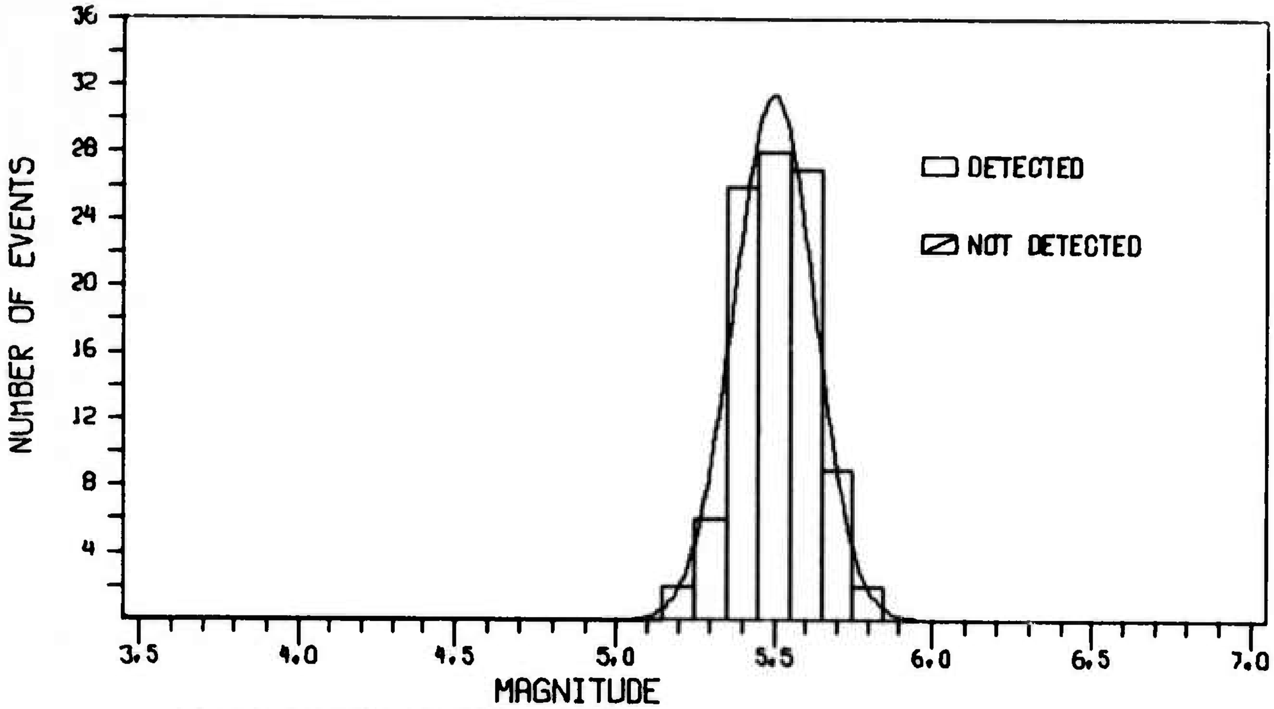
$$0.25 \leq \sigma \leq 0.60 . \tag{III-3}$$

(We recall that the true value of σ is 0.40.) Restricting σ in this way should not make a significant difference in practical applications of the method, since only an approximate a priori knowledge of signal variance is required.

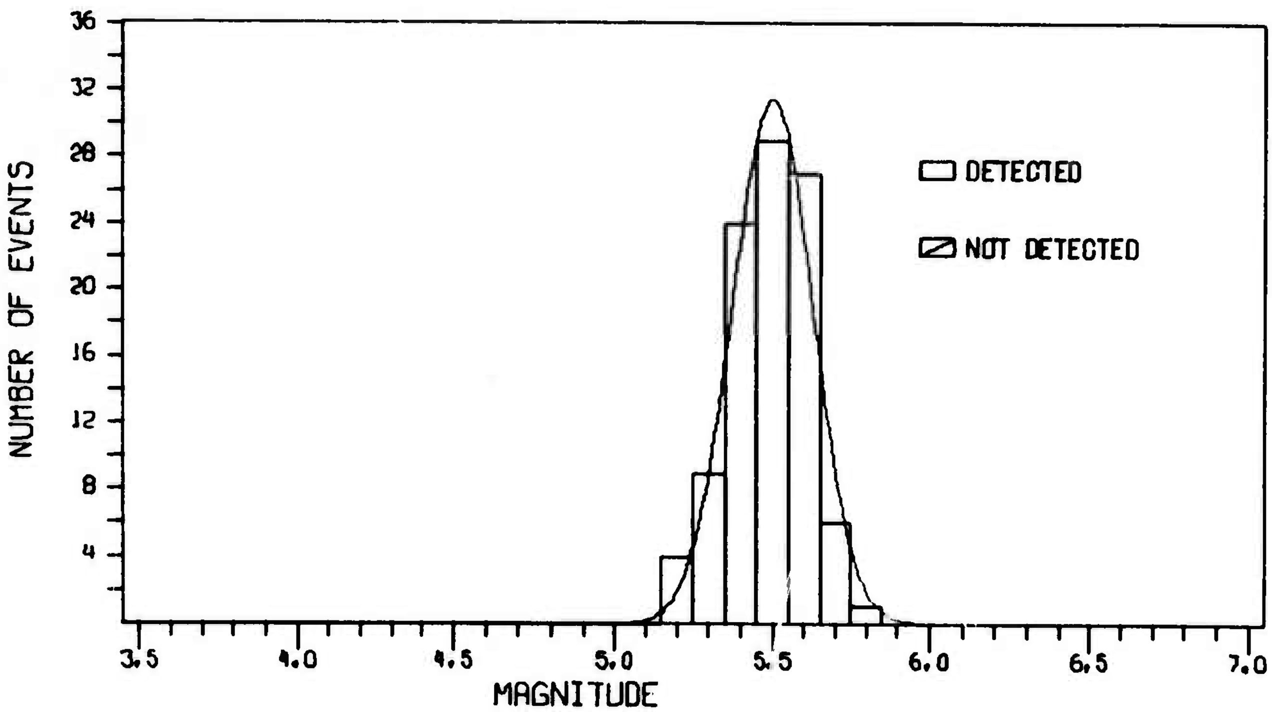
Results from these simulations for Network 1 are presented in Figures III-8 through III-11 for event magnitudes $\mu = 5.5$, 5.0, 4.5, and 4.0, respectively. The following major points may be made:

- For an event of magnitude 5.5, no significant difference is seen compared to the case of known σ (Figure III-1). This is consistent with the observation that the maximum likelihood estimates of μ and σ are independent if all stations detect
- For $\mu = 5.0$, 4.5, and 4.0, the maximum likelihood estimates are significantly better than the conventional estimates. Furthermore, it appears that the former estimates are only slightly inferior to those made with a known value of σ, as was shown in Figures III-2 through III-4
- As in the case of known σ, the Cramer-Rao bounds appear to give an adequate picture of the variance of the maximum likelihood estimator, although we only developed those bounds for the case of a known σ.

As a final part of the simulation experiment, we investigated the consequence of deliberately using a wrong value of σ when maximizing the



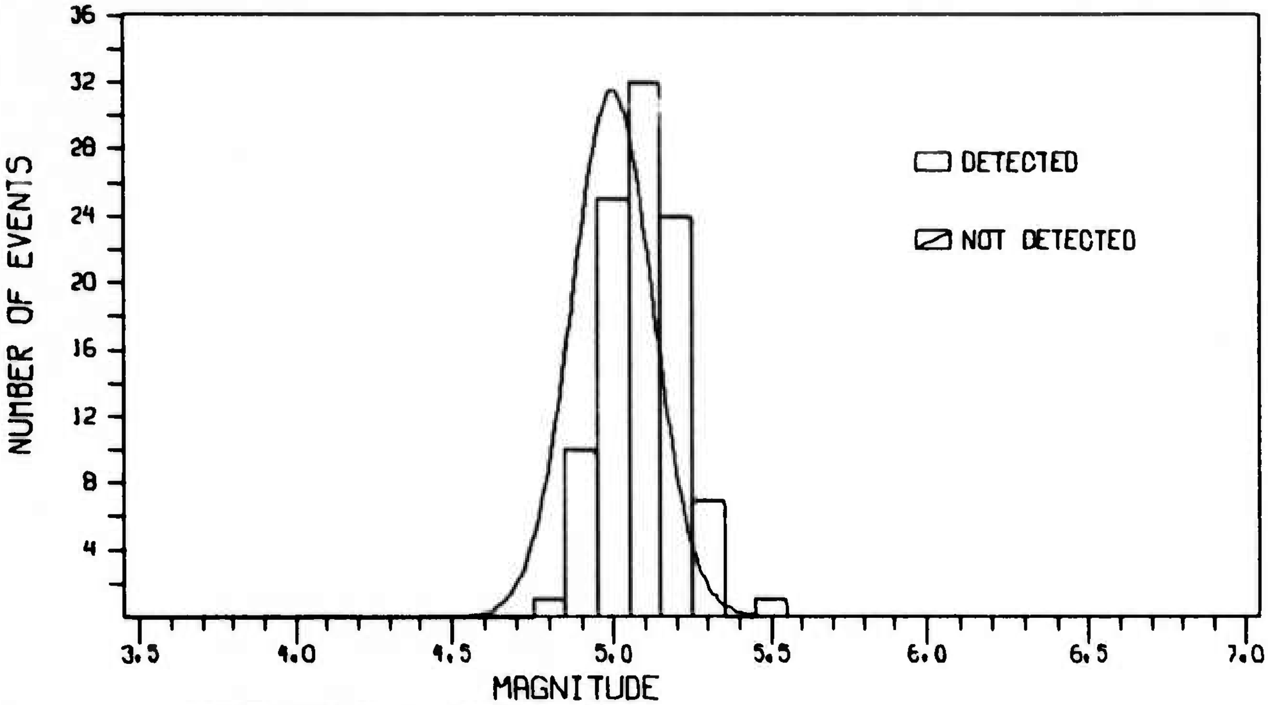
EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES



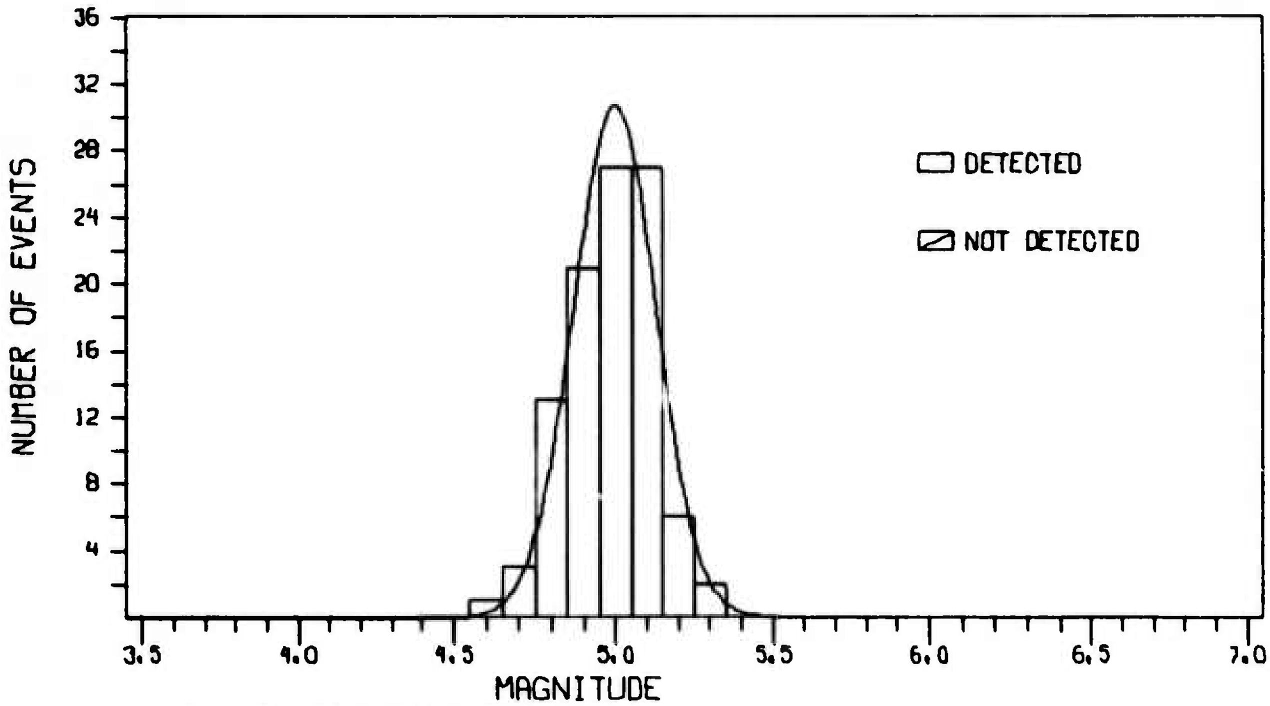
EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (0.25<SIGMA<0.60)

FIGURE III-8

SIMULATED PERFORMANCE OF NETWORK 1 FOR 100 EVENTS OF MAGNITUDE 5.5 AND UNKNOWN STANDARD DEVIATION



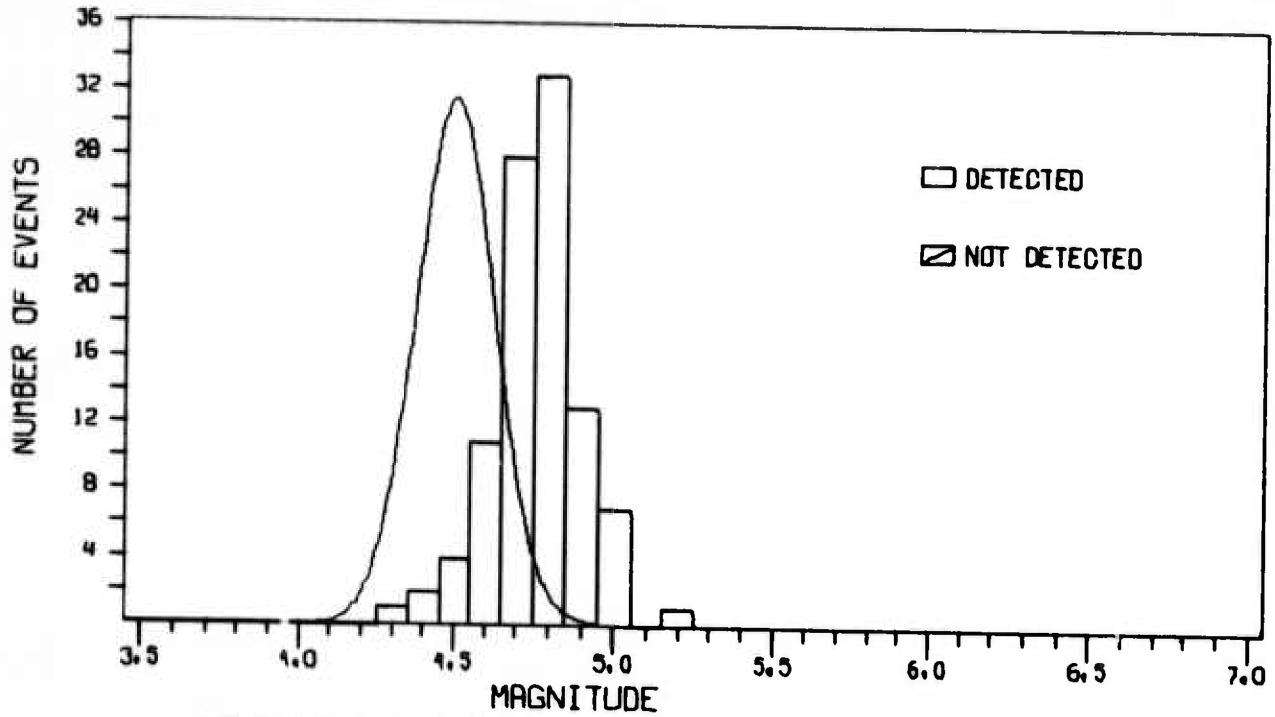
EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES



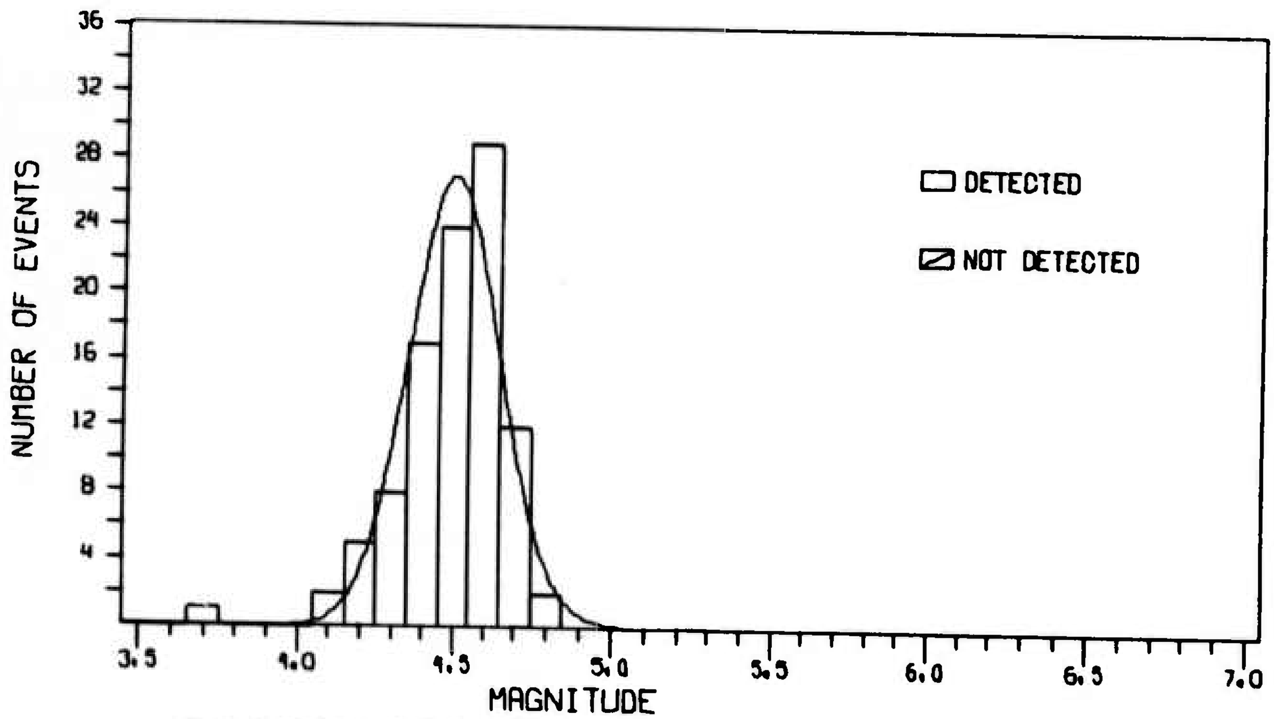
EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (0.25<SIGMA<0.60)

FIGURE III-9

SIMULATED PERFORMANCE OF NETWORK 1 FOR 100 EVENTS OF MAGNITUDE 5.0 AND UNKNOWN STANDARD DEVIATION



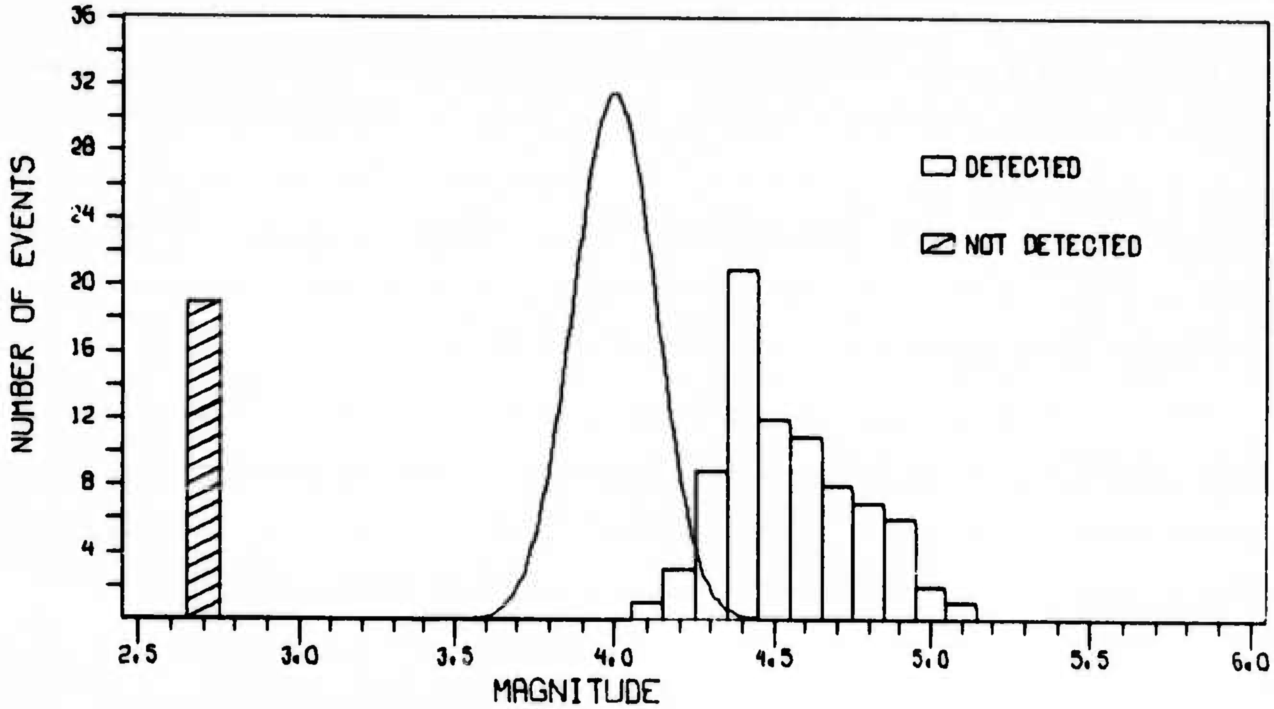
EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES



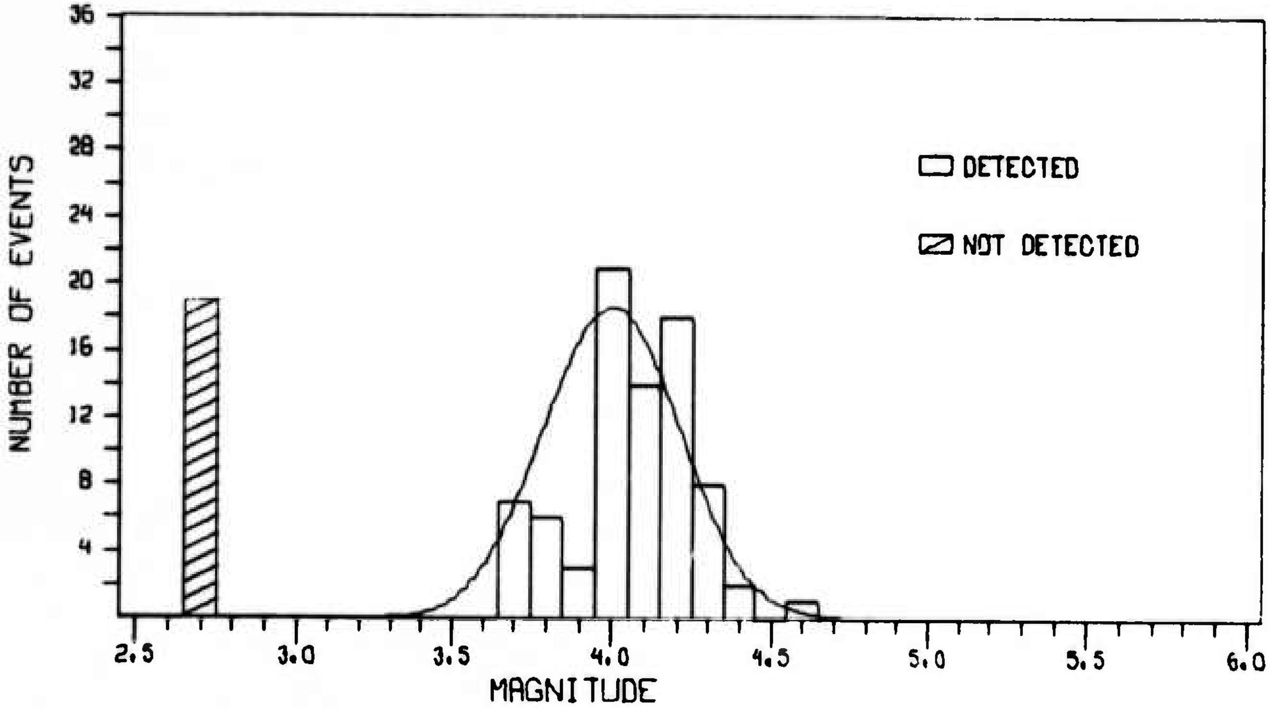
EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (0.25<SIGMA<0.60)

FIGURE III-10

SIMULATED PERFORMANCE OF NETWORK 1 FOR 100 EVENTS OF MAGNITUDE 4.5 AND UNKNOWN STANDARD DEVIATION



EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES - CASE 3B



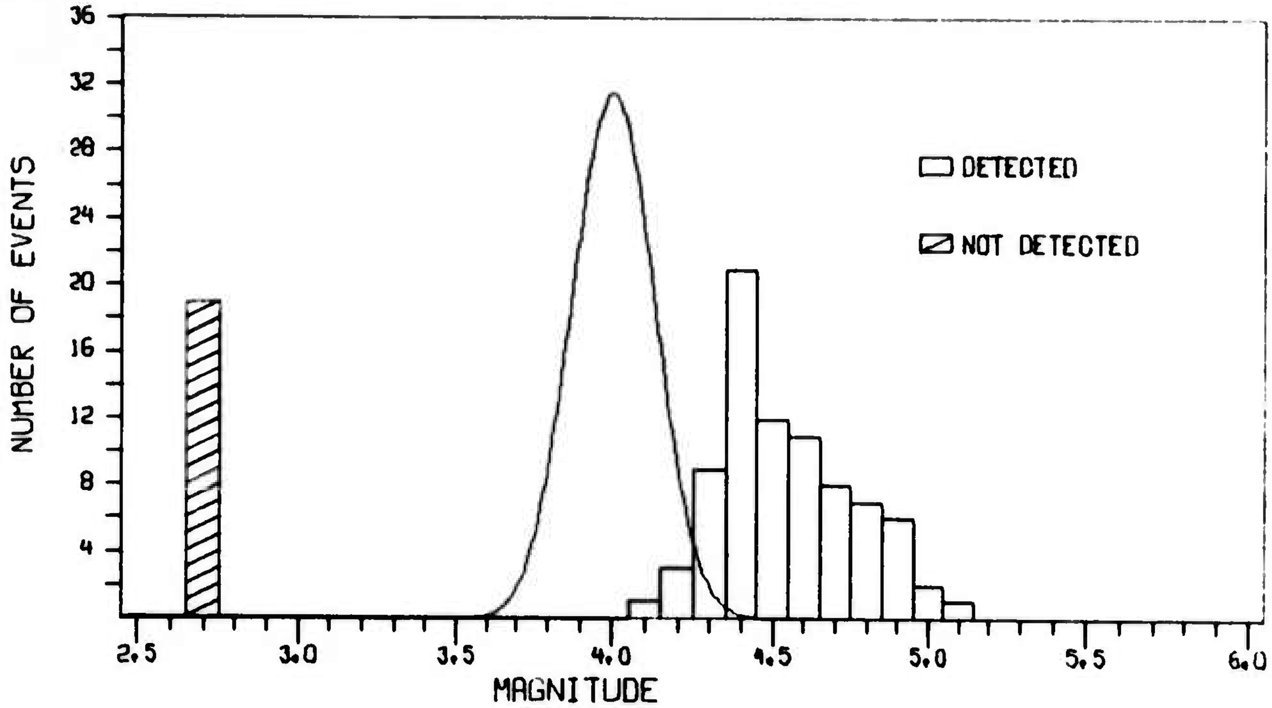
EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (0.25<SIGMA<0.60)

FIGURE III-11

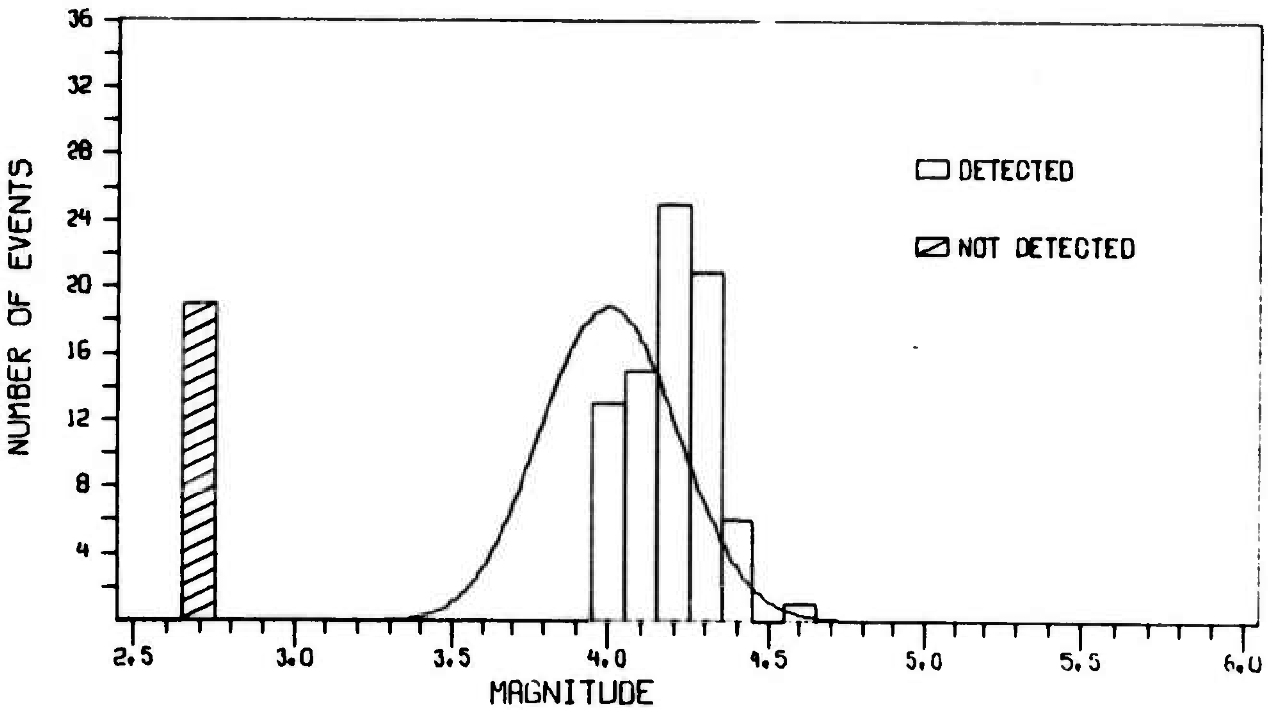
SIMULATED PERFORMANCE OF NETWORK 1 FOR 100 EVENTS OF MAGNITUDE 4.0 AND UNKNOWN STANDARD DEVIATION

likelihood function (II-5). The effects of such a mistake will be most pronounced in those cases when few stations detect. Figures III-12 and III-13 show the resulting estimates for $\mu = 4.0$, $\sigma = 0.4$ as simulation parameters, with σ set to 0.25 and 0.60, respectively in the estimation process. A definite bias is seen in both cases, although the maximum likelihood estimates are still more accurate than the conventional estimates.

From this last result we conclude that, unless σ is known with good confidence, the best way in practice to apply the maximum likelihood estimation method is to use a two-parameter maximization technique, and allow σ to vary within predefined, reasonable bounds.



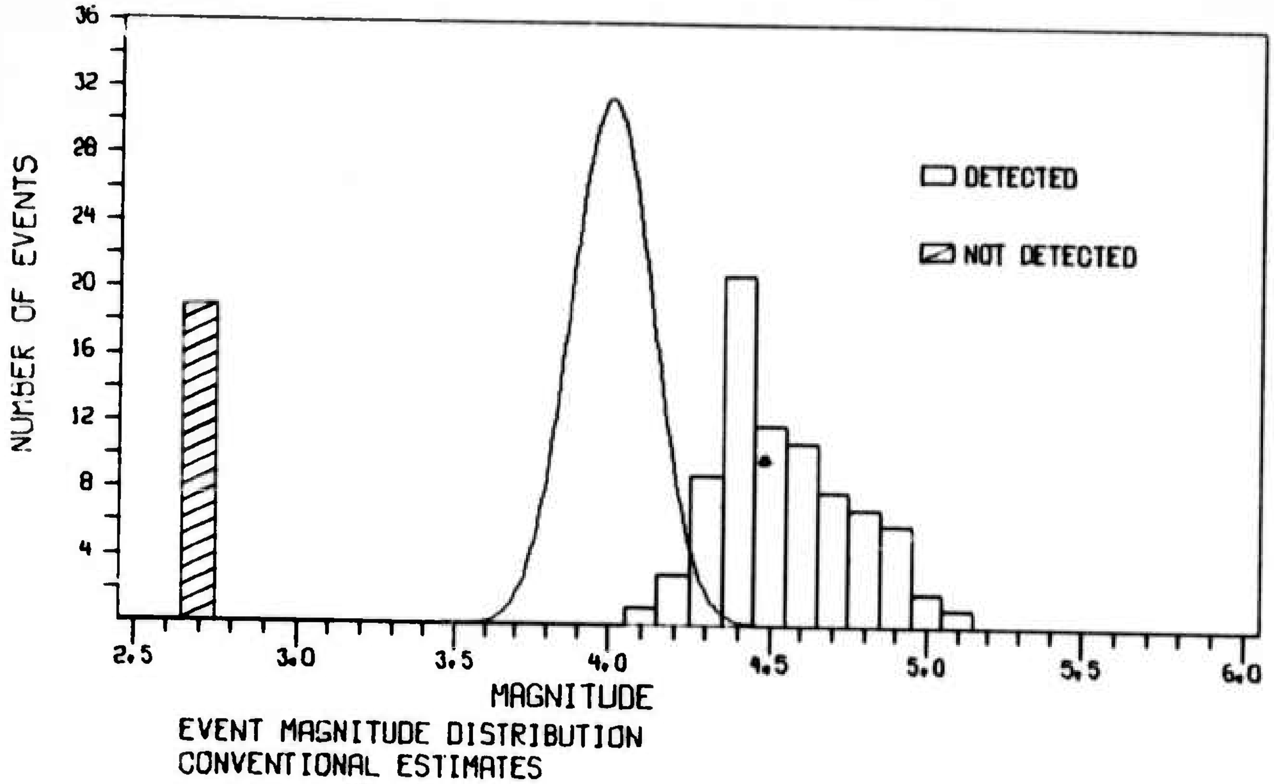
EVENT MAGNITUDE DISTRIBUTION
CONVENTIONAL ESTIMATES



EVENT MAGNITUDE DISTRIBUTION
MAX LIKE ESTIMATES (SIGMA=0.25)

FIGURE III-12

SIMULATED PERFORMANCE OF MAXIMUM LIKELIHOOD ESTIMATOR WHEN TOO LOW STANDARD DEVIATION IS APPLIED



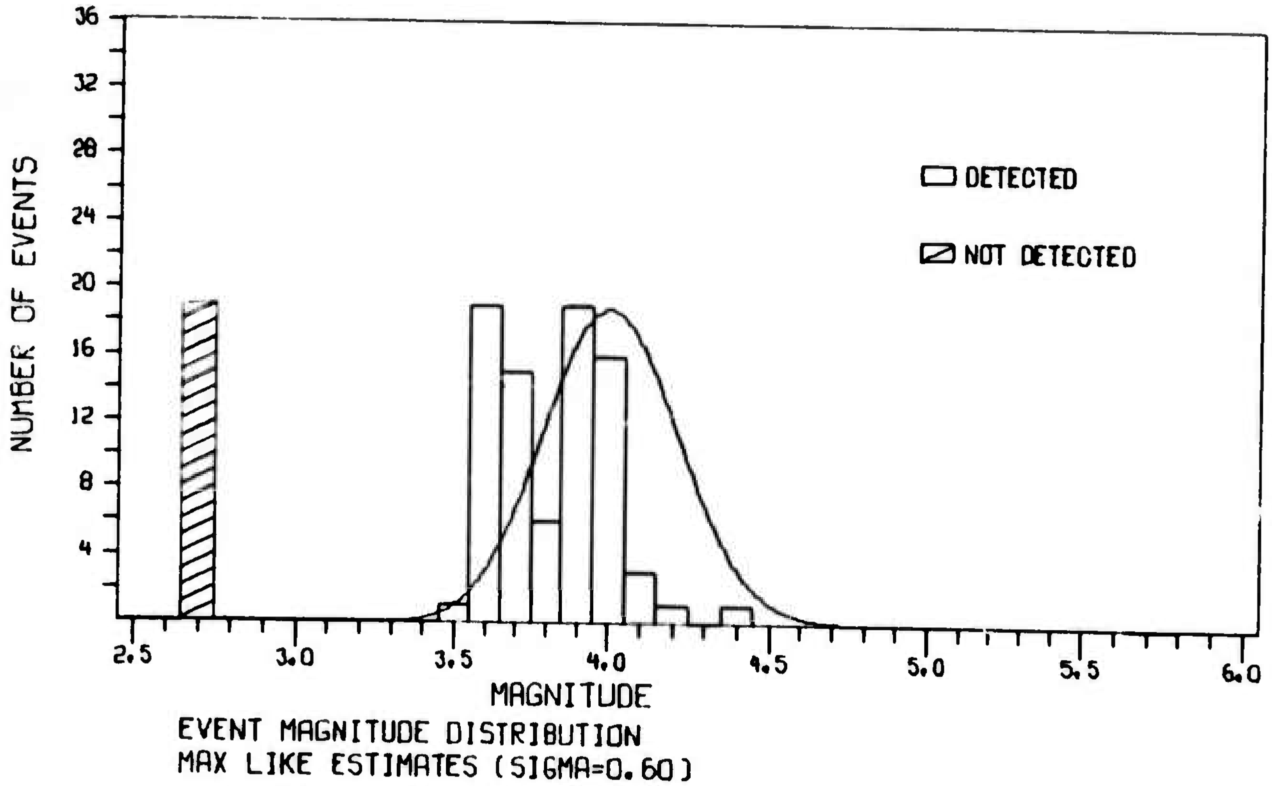


FIGURE III-13

SIMULATED PERFORMANCE OF MAXIMUM LIKELIHOOD ESTIMATOR WHEN TOO HIGH STANDARD DEVIATION IS APPLIED

SECTION IV
DATA ANALYSIS

In this section the maximum likelihood network magnitude estimation technique is applied to actual seismic data, and the results compared to conventional estimates. Two cases are discussed, one concerning m_b estimates by a subnet of the World-Wide Standard Seismograph Network (WWSSN), the other relating to Rayleigh wave magnitudes estimated by the Very Long Period Experiment (VLPE) network.

A. APPLICATION TO A SUBSET OF WWSSN STATIONS

In order to test the effectiveness of the maximum likelihood method in estimating earthquake bodywave magnitudes, the natural approach was to apply the method to data from the World-Wide Standard Seismograph Network (WWSSN). Unfortunately, a number of stations in this network do not routinely report amplitude and period with each detection. Consequently a very limited number of magnitude estimates are available, and stations not reporting magnitude cannot automatically be classified as not detecting. Furthermore, seismic noise levels are not reported by these stations, and must be estimated.

The following approach was undertaken in order to minimize these problems:

- First of all we selected as a reference event set an earthquake aftershock sequence (from the Tadzhik-Sinkiang border region, August 11-31, 1974). This provided a large number of events (about 60) from the same seismic region, and within a relatively short time span, during which the WWSSN could be assumed to be fairly constant

- Secondly, a subset of 13 WWSSN stations was selected. These stations, which are listed in Table IV-1, were the ones that appeared to report the largest number of event magnitudes from the aftershock sequence. The large arrays NORSAR and LASA were deliberately not included in this subset

- For each of the above 13 stations, we assumed a 'magnitude reporting threshold' to be the average value of the three lowest magnitudes actually reported. This provided a set of numbers (Table IV-1) which were used as threshold values (a_i, $i = 1, 2, \ldots, 13$) in the maximum likelihood estimation.

We thereby obtained a reasonably homogeneous network with threshold magnitudes ranging from $m_b = 4.3$ to 5.3. The network had a balanced distribution of epicentral distances, and all stations were located in the teleseismic range of 30-90 degrees. The azimuthal distribution was not quite so good, but was still thought to be adequate. By excluding NORSAR from the network, we were able to use magnitudes from this station as a reference to check the validity of our network estimates.

Table IV-2 lists the events of the reference set. All events reported in the PDE bulletins for this aftershock sequence within the time interval August 11- August 31, 1974 are included, except for one event occurring while NORSAR was out of operation and three events that were not detected by any of our 13 stations. Individual station magnitudes are included in the table, as well as network magnitudes estimated both by conventional averaging and by the maximum likelihood technique.

For the maximum likelihood method, equations (II-8) and (II-9) were applied, with σ preset to 0.40 and $\sigma_{Ti} = 0.2$ ($i = 1, 2, \ldots, 13$). The choice of σ is consistent with the results by Veith and Clawson (1972), while σ_{Ti} mainly reflects variations in seismic noise level, and can therefore reasonably be set to a small value for the aftershock sequence.

TABLE IV-1

LIST OF SELECTED WWSSN STATIONS AND THEIR LOCATION RELATIVE TO A SELECTED EARTHQUAKE AFTERSHOCK SEQUENCE

Station Number	Name	Latitude	Longitude	Distance	Azimuth	Threshold
1	NUR	60.5N	24.5E	36.8	321	4.7
2	BRG	50.9N	14 E	42.6	306	4.6
3	CLL	51.3N	13 E	43.1	307	4.7
4	MAT	36.5N	138.2E	49.8	72	4.8
5	BNG	4.4N	18.5E	60.8	250	4.4
6	MBC	76.2N	119.3W	64.3	4	4.8
7	INK	68.3N	133.5W	70.7	10	5.3
8	GIL	65.0N	147.5W	71.1	17	4.6
9	BUL	20.1S	28.6E	72.7	224	4.5
10	PMR	61.6N	149.1W	73.5	20	4.3
11	BLC	64.3N	96 W	76.3	355	4.9
12	YKC	62.5N	114.5W	78.2	4	4.9
13	FFC	54.7N	102.0W	86.2	358	4.8
REF	NORSAR	60.8N	10.8E	43.6	320	

TABLE IV-2

SELECTED WWSSN STATION MAGNITUDES AND NETWORK MAGNITUDE ESTIMATES FOR AN EARTHQUAKE AFTERSHOCK SEQUENCE FROM THE TADZHIK-SINKIANG BORDER REGION AUGUST 11-31, 1974
(PAGE 1 OF 3)

EVENT NO	DATE	TIME	INDIVIDUAL STATION MAGNITUDES 1	2	3	4	5	6	7	8	9	10	11	12	13	AVG MB	M-L MB	NCFSAR MB
1	8/11	02.29.48	-	-	-	-	-	5.3	-	5.0	4.9	4.3	-	-	-	4.9	4.5	4.3
2	8/11	02.37.11	4.8	-	-	-	5.2	5.7	-	5.0	4.9	4.5	5.2	5.0	5.2	5.1	4.9	4.9
3	8/11	03.05.11	-	-	-	-	4.6	5.2	-	4.7	-	-	-	-	-	4.8	4.6	3.9
4	8/11	03.24.11	-	-	-	-	-	4.8	-	-	-	-	-	-	-	4.8	4.0	3.9
5	8/11	04.28.46	5.0	4.8	4.9	5.0	5.1	5.7	-	4.9	4.6	5.0	5.1	5.2	5.3	5.1	5.0	4.7
6	8/11	05.12.33	5.3	5.2	5.3	5.7	5.9	5.9	5.6	5.8	5.2	5.3	5.3	5.3	5.8	5.5	5.5	5.0
7	8/11	05.19.33	5.1	-	5.0	5.0	-	6.0	5.6	5.1	4.9	4.9	5.2	5.2	5.3	5.2	5.1	4.9
8	8/11	05.23.52	5.9	5.6	5.6	5.5	-	6.1	5.9	-	5.6	5.5	5.7	5.6	5.7	5.7	5.5	5.6
9	8/11	05.33.48	-	-	-	-	-	5.3	-	-	4.6	4.5	-	-	-	4.8	4.4	4.6
10	8/11	07.02.08	5.1	-	5.2	5.5	4.9	5.9	5.4	5.3	5.0	4.8	5.2	5.2	5.4	5.2	5.2	5.2
11	8/11	08.02.54	4.8	4.7	4.8	5.1	5.3	5.8	-	5.4	5.2	5.0	-	5.1	5.2	5.1	5.1	4.7
12	8/11	09.08.58	4.9	4.7	4.9	5.0	5.5	5.9	-	5.2	5.3	4.9	5.1	5.1	-	5.1	5.1	4.7
13	8/11	12.45.04	4.9	-	-	-	4.8	5.1	-	-	-	-	-	-	-	4.9	4.4	4.6
14	8/11	13.21.17	-	-	-	-	-	5.1	-	-	-	-	-	-	-	5.1	4.1	4.1
15	8/11	13.38.24	-	-	-	-	-	5.2	-	4.9	-	-	-	-	-	5.1	4.3	4.2
16	8/11	13.59.23	-	-	-	-	-	5.2	-	4.9	4.6	-	-	-	-	4.9	4.4	4.1
17	8/11	17.40.43	-	-	-	-	4.6	4.8	-	-	-	-	-	-	-	4.7	4.2	4.2
18	8/11	19.30.36	-	-	-	-	4.8	5.0	-	4.4	-	-	-	-	-	4.7	4.3	4.5
19	8/11	20.05.30	5.9	5.6	5.4	6.3	5.8	6.3	6.0	5.8	6.0	5.5	5.7	6.0	6.3	5.9	5.9	5.4
20	8/11	21.21.33	-	5.7	5.3	6.4	5.9	5.9	5.9	6.2	5.6	5.8	5.8	6.1	6.1	5.9	5.8	5.4
21	8/11	21.50.07	-	-	4.7	-	4.6	5.2	-	4.8	-	-	-	-	5.1	4.9	4.6	4.4

TABLE IV-2

SELECTED WWSSN STATION MAGNITUDES AND NETWORK MAGNITUDE ESTIMATES FOR AN EARTHQUAKE AFTERSHOCK SEQUENCE FROM THE TADZHIK-SINKIANG BORDER REGION AUGUST 11-31, 1974

(PAGE 2 OF 3)

EVENT NO	DATE	TIME	INDIVIDUAL STATION MAGNITUDES 1	2	3	4	5	6	7	8	9	10	11	12	13	AVG MB	M-L MB	NORSAR MB
22	8/11	21.56.29	-	-	-	-	4.8	5.4	-	4.7	-	-	-	-	-	5.0	4.4	4.2
23	8/11	22.10.27	-	-	-	-	4.4	5.1	-	-	-	-	-	-	-	4.8	4.2	4.3
24	8/11	23.18.58	5.1	4.7	4.7	-	4.8	5.6	-	5.3	-	-	5.2	4.8	5.2	5.0	4.9	5.0
25	8/12	13.40.10	-	-	-	-	-	5.0	-	-	-	-	-	-	-	5.0	4.1	3.8
26	8/12	14.14.54	-	-	-	4.8	-	5.4	-	-	-	-	-	-	-	5.1	4.3	4.2
27	8/12	21.17.47	4.7	4.8	5.0	5.2	5.4	5.9	-	5.1	4.8	-	4.9	5.1	5.4	5.1	5.0	4.5
28	8/12	21.57.17	4.8	-	-	-	5.0	5.8	5.3	4.9	-	-	4.9	5.0	-	5.1	4.8	4.6
29	8/12	22.44.35	-	-	-	-	4.4	-	-	-	-	-	-	-	-	4.4	3.9	4.2
30	8/13	11.14.39	-	-	-	-	4.5	5.2	-	-	-	-	-	-	-	4.9	4.3	4.2
31	8/13	14.16.06	-	-	-	-	-	5.2	-	-	4.7	-	-	-	-	5.0	4.3	4.2
32	8/13	14.48.13	-	-	-	-	-	4.9	-	-	-	-	-	-	-	4.9	4.1	3.8
33	8/13	21.19.17	-	-	-	-	4.5	5.5	-	-	-	-	-	-	-	5.0	4.3	4.2
34	8/14	01.31.02	-	-	-	-	4.5	5.0	-	-	-	-	-	-	-	4.8	4.2	4.0
35	8/14	06.18.20	-	-	-	-	-	5.0	-	-	-	-	-	-	-	5.0	4.1	3.8
36	8/14	07.50.56	-	-	-	4.8	4.6	4.9	-	-	-	-	-	-	-	4.8	4.3	4.5
37	8/14	22.06.52	4.9	4.6	-	4.8	4.9	-	5.3	5.3	-	-	5.1	-	-	5.0	4.7	4.7
38	8/15	04.38.39	4.9	-	-	-	4.5	5.3	-	-	-	-	-	-	-	4.9	4.4	4.7
39	8/15	11.22.48	-	-	-	-	4.5	5.2	-	-	-	-	-	-	-	4.9	4.3	3.9
40	8/16	00.11.08	5.0	4.6	4.8	5.0	-	5.4	-	-	-	-	-	4.9	-	5.0	4.6	4.6
41	8/17	03.23.08	-	-	-	-	-	5.0	-	-	-	-	-	-	-	5.0	4.1	4.1
42	8/17	23.50.58	5.0	-	-	5.0	4.9	5.8	-	-	-	5.2	-	5.1	-	5.2	4.6	4.2

TABLE IV-2

SELECTED WWSSN STATION MAGNITUDES AND NETWORK MAGNITUDE ESTIMATES FOR AN EARTHQUAKE AFTERSHOCK SEQUENCE FROM THE TADZHIK-SINKIANG BORDER REGION AUGUST 11-31, 1974
(PAGE 3 OF 3)

EVENT NO	DATE	TIME	INDIVIDUAL STATION MAGNITUDES 1	2	3	4	5	6	7	8	9	10	11	12	13	AVG MB	M-L MB	NCRSAF MB
43	8/18	16.03.18	-	-	-	-	4.7	-	-	4.9	-	-	-	-	-	4.8	4.3	4.4
44	8/20	10.18.43	-	-	-	-	-	5.1	-	-	-	-	-	-	-	5.1	4.1	3.9
45	8/20	15.14.36	-	-	-	-	4.7	4.9	-	-	-	-	-	-	-	4.8	4.2	4.0
46	8/20	16.42.25	-	-	-	4.7	4.9	-	-	-	-	-	-	-	-	4.8	4.2	4.1
47	8/20	17.33.00	-	-	-	-	5.0	-	-	-	-	-	-	-	-	5.0	4.1	4.1
48	8/21	18.08.29	4.7	-	-	-	4.8	5.3	-	4.9	4.5	-	-	-	-	4.8	4.5	4.4
49	8/21	18.45.16	5.1	4.6	-	-	5.2	5.5	-	5.3	4.9	-	-	4.9	4.8	5.0	4.8	4.9
50	8/22	09.16.48	-	-	-	-	-	4.9	-	-	-	-	-	-	-	4.9	4.1	4.0
51	8/23	16.26.30	4.9	-	-	4.9	4.7	5.1	-	-	4.6	-	-	-	-	4.8	4.5	4.5
52	8/24	11.21.46	4.9	4.5	4.7	5.1	4.9	5.5	-	5.0	-	4.5	4.9	-	-	4.9	4.8	4.7
53	8/24	12.14.37	-	-	-	-	4.7	5.8	-	5.0	4.8	4.3	4.9	-	4.8	4.9	4.7	4.4
54	8/25	17.41.18	-	-	-	-	4.5	4.9	-	-	-	-	-	-	-	4.7	4.2	3.6
55	8/26	10.22.28	-	-	-	-	4.4	4.8	-	-	-	-	-	-	-	4.6	4.2	4.1
56	8/27	05.43.33	4.8	-	-	-	-	-	-	4.9	-	-	-	-	-	4.9	4.3	4.5
57	8/27	17.33.58	5.2	4.9	-	5.6	-	-	-	5.7	-	-	-	5.1	5.4	5.3	4.9	4.9
58	8/28	09.22.07	-	-	-	-	-	5.0	-	-	4.5	-	-	-	-	4.8	4.2	4.3
59	8/28	17.05.03	-	-	-	-	-	-	-	4.7	-	-	-	-	-	4.7	4.0	3.9
60	8/29	08.25.20	-	-	-	-	-	5.3	-	-	-	-	-	-	-	5.3	4.2	4.1
61	8/31	17.13.55	-	-	-	-	-	5.1	-	-	-	-	-	-	-	5.1	4.1	4.2

Plots of the estimated network magnitudes versus the NORSAR m_b values are shown in Figures IV-1 and IV-2 for the conventional method and the maximum likelihood method, respectively. It is clear from these plots that the latter method gives by far the best correspondence between the network and NORSAR, and it is concluded that the maximum likelihood method produces more reliable magnitudes than the conventional method for this data set. The bias of about 0.1 m_b units in Figure IV-2 is probably caused by the beamforming loss at NORSAR, which is not compensated for in NORSAR m_b estimates.

It is observed from Figure IV-1 that the conventional estimates seem to saturate toward a magnitude of about 5.0 as the NORSAR m_b value decreases. This is precisely what could be expected from our simulation results in Section III. The maximum likelihood estimates will also reach a saturation level for any given network, but this level is at least half a magnitude unit lower for the particular situation discussed here.

It is appropriate to add that we also attempted to apply the maximum likelihood method with other parameter settings such as $\sigma_{Ti} = 0$ for $i = 1, 2, \ldots, n$ and a two parameter maximization of the likelihood function. The results were essentially the same as the ones presented, and therefore indicate a robustness of the maximum likelihood method which was also demonstrated in the simulation experiments in Section III.

As a final remark, we note that some of the largest events were not measured at one or two stations; apparently due to high coda level remaining from a previous large event. Although these non-detections did not substantially impact the maximum likelihood magnitude estimates, it is nevertheless clear that the assumed Gaussian distribution of threshold magnitudes does not adequately represent such cases. Therefore, we feel that it is very much preferrable to actually measure the threshold magnitude for each non-detection if at all possible.

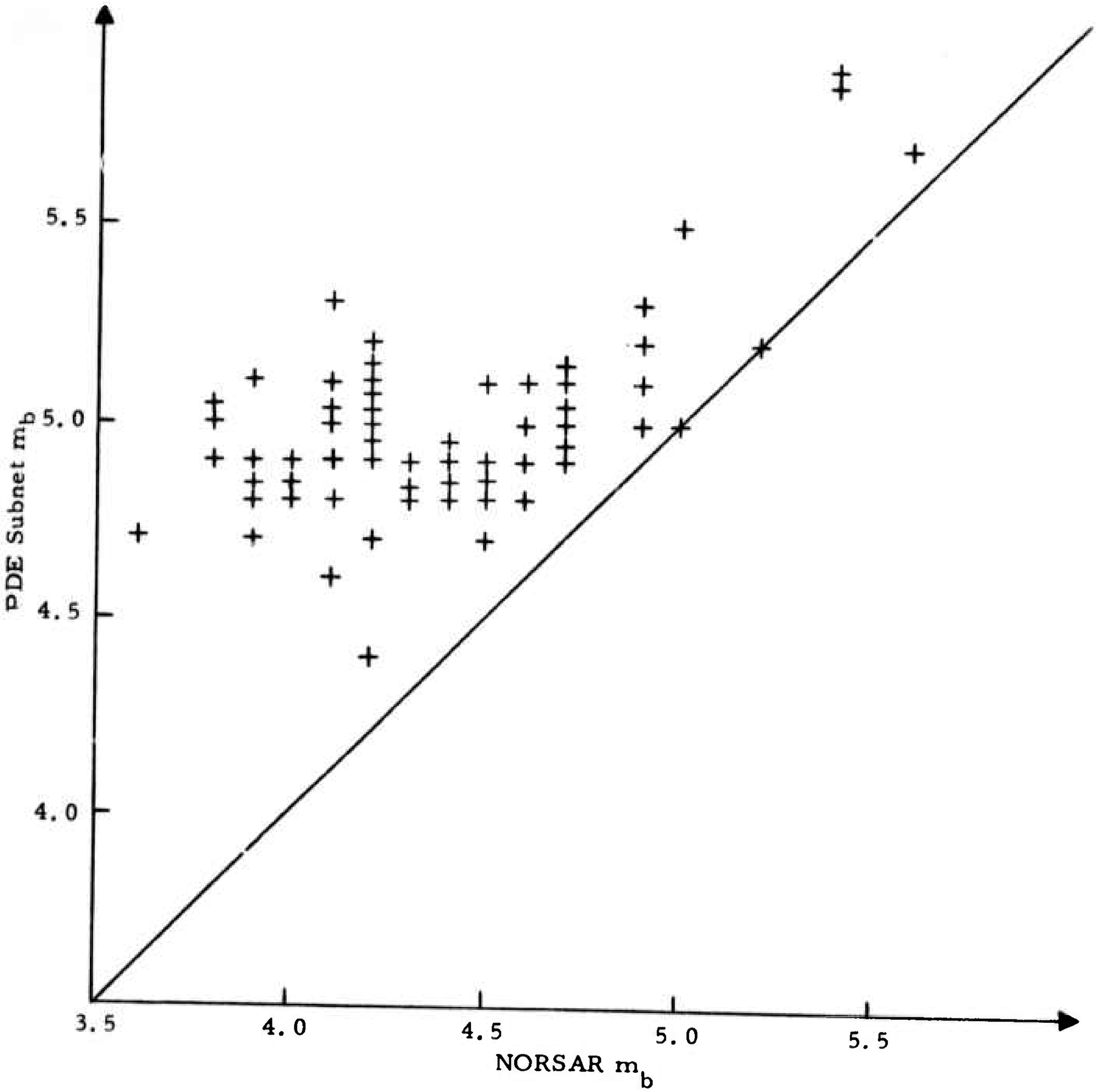


FIGURE IV-1

WWSSN SUBNET m_b VERSUS NORSAR m_b FOR EVENTS FROM AN AFTERSHOCK SEQUENCE USING CONVENTIONAL NETWORK MAGNITUDE ESTIMATES

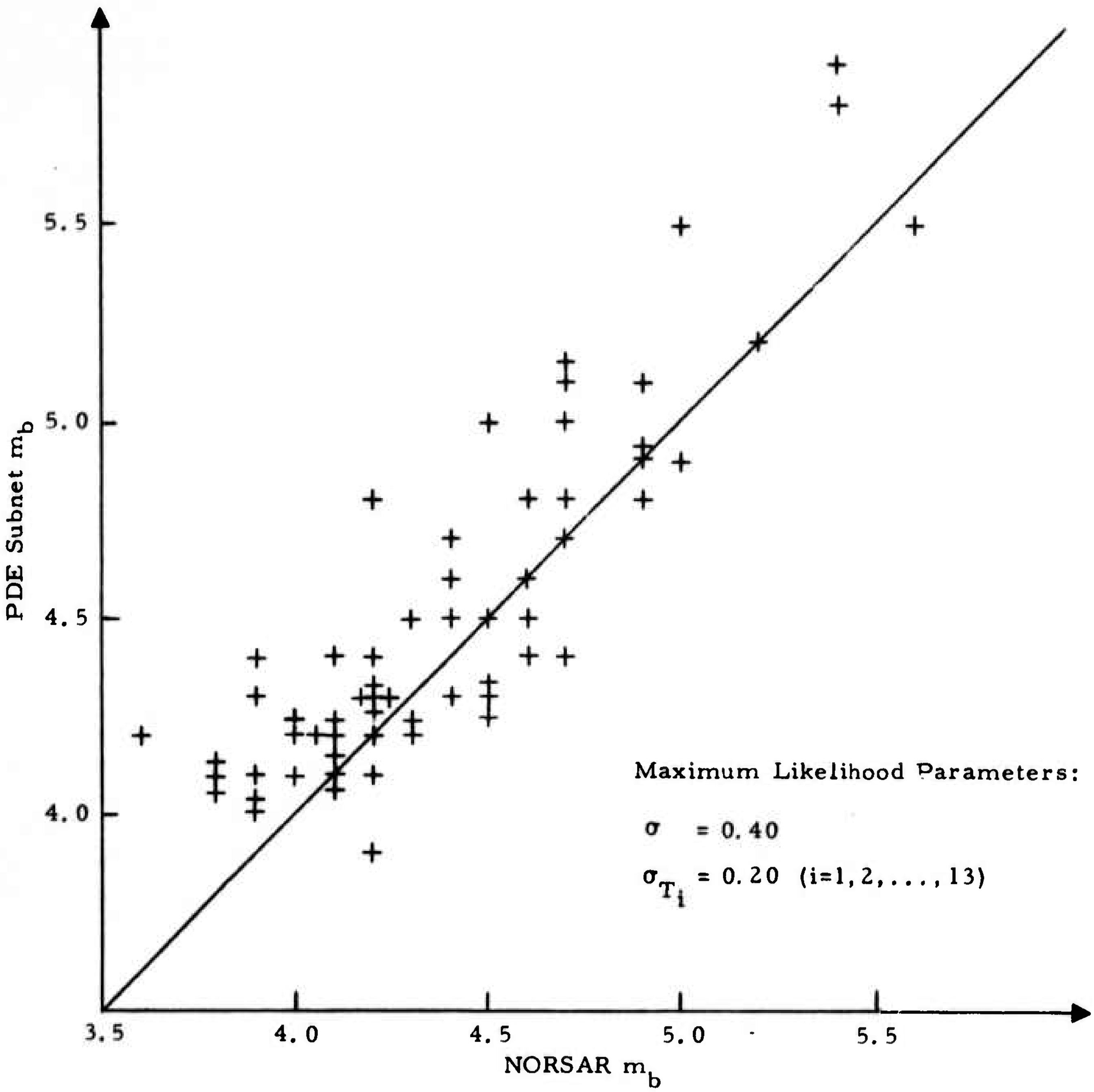


FIGURE IV-2

WWSSN SUBNET m_b VERSUS NORSAR m_b FOR EVENTS FROM AN AFTERSHOCK SEQUENCE USING MAXIMUM LIKELIHOOD NETWORK MAGNITUDE ESTIMATES

B. APPLICATION TO THE VLPE NETWORK

Data from the Very Long Period Experiment (VLPE) network have recently been extensively analyzed by Lambert et al. (1974). Their data base consisted of more than 1000 seismic events from Eurasia during 1972, 1973, and 1974. The 11 individual stations of the VLPE network are listed in Table IV-3. None of these stations were operational throughout the full time period covered, and, most often, data for a given event were available only from five or fewer stations. Hence the VLPE network was not ideal for our purpose, but was still found useful to illustrate some aspects of the maximum likelihood estimation technique.

Our procedure in processing the VLPE data was as follows:

- Individual VLPE station Rayleigh wave data were obtained from all events in Lambert's data base that were not classified as presumed explosions
- For each event, a subset of stations was selected by eliminating all stations that either were non-operational, had poor data quality such as spikes, or were influenced by the coda level of a previous large event
- The resulting sub-network for each event was split into detecting and non-detecting stations. For each detecting station, the Rayleigh wave magnitude measured by Lambert et al. was assigned. This usually was the 20 second period M_s. If no 20 second magnitude had been measured, M_s at 30 or 40 seconds periods were used instead
- For each event in this data base, M_s values were estimated by conventional averaging over all detecting stations and also by applying maximum likelihood estimation

TABLE IV-3

VERY LONG PERIOD EXPERIMENT (VLPE) STATIONS, LOCATIONS AND ESTIMATED DETECTION THRESHOLDS

Station Number	Station Name	Designator	Latitude	Longitude	Threshold at 60 Degrees $\overline{a}_i$
1	Charters Towers, Australia	CTA	20.09S	146.26E	3.9
2	Chiang Mai, Thailand	CHG	18.79N	98.98E	3.6
3	Fairbanks, Alasks	FBK	64.90N	148.01W	3.9
4	Toledo, Spain	TLO	39.86N	4.02W	3.7
5	Eilat, Israel	EIL	29.55N	34.95E	3.9
6	Kongsberg, Norway	KON	59.65N	9.59E	3.6
7	Ogdensburg, New Jersey	OGD	41.07N	74.62W	3.5
8	Kipapa, Hawaii	KIP	21.42N	158.02W	3.6
9	Albuquerque, New Mexico	ALQ	34.94N	106.46W	3.8
10	La Paz, Bolivia	ZLP	16.50S	68.13W	3.7
11	Matsushiro, Japan	MAT	36.54N	138.21E	3.6

- In the maximum likelihood estimation, the approach given by (II-8) and (II-9) was utilized. A signal standard deviation of $\sigma = 0.40$ was used, together with values of $\sigma_{Ti} = 0.40$ $(i = 1, 2, \ldots, 11)$. These last values are consistent with observed variations in noise level at the VLPE sites (Prahl, 1974), while the value of σ was chosen somewhat arbitrarily

- The actual detection threshold estimates at each individual station $(\bar{a}_i)$ at 60 degrees distance are listed in Table IV-3. These values were obtained from Lambert et al., 1974, using the 50 percent M_s detection threshold of each station and compensating for the average epicentral distance to the event set. In our maximum likelihood estimation model, the actual station threshold a_i for a given event was found as:

  $$a_i = \bar{a}_i - \log 60 + \log \Delta \qquad \text{(IV-1)}$$

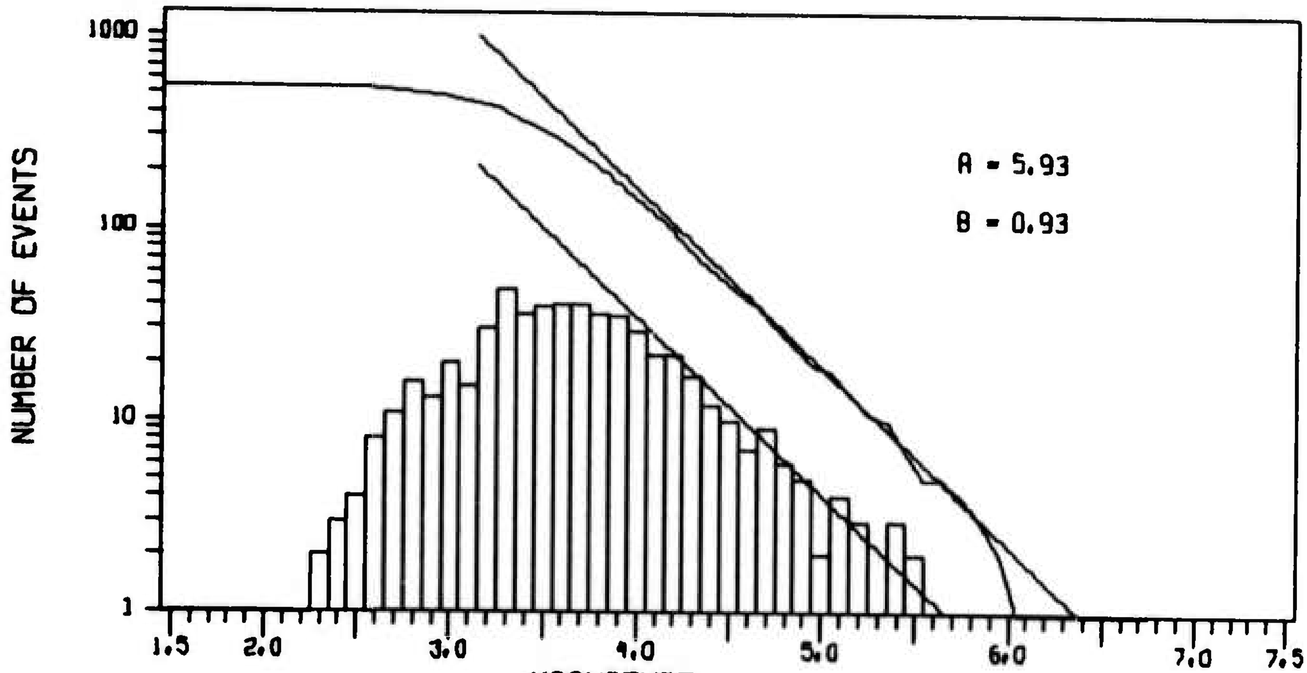
  where Δ is the epicentral distance from the station to the event in degrees.

We refer to Lambert et al. (1974) for a comprehensive list of individual station data and event parameters for the VLPE event set.

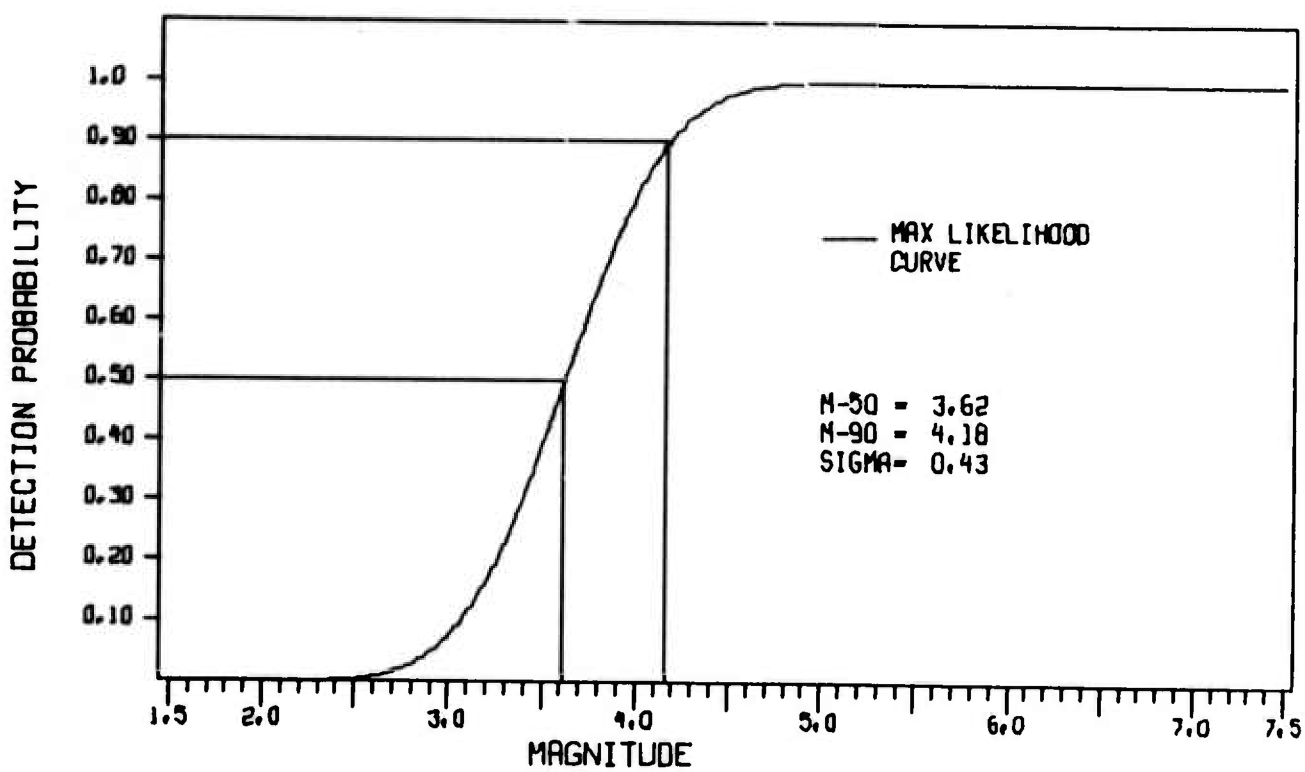
Figures IV-3 and IV-4 give a comparison between the results from the two methods of estimation. Each of these figures shows the seismicity line and the incremental detection curve for the event data set, and has been derived by applying the method of Lacoss and Kelly (1969). The seismicity line is defined by the magnitude frequency relationship:

$$\log_{10} N_c = A - B \cdot M \qquad \text{(IV-2)}$$

where N_c is the number of events with surface-wave magnitude exceeding a



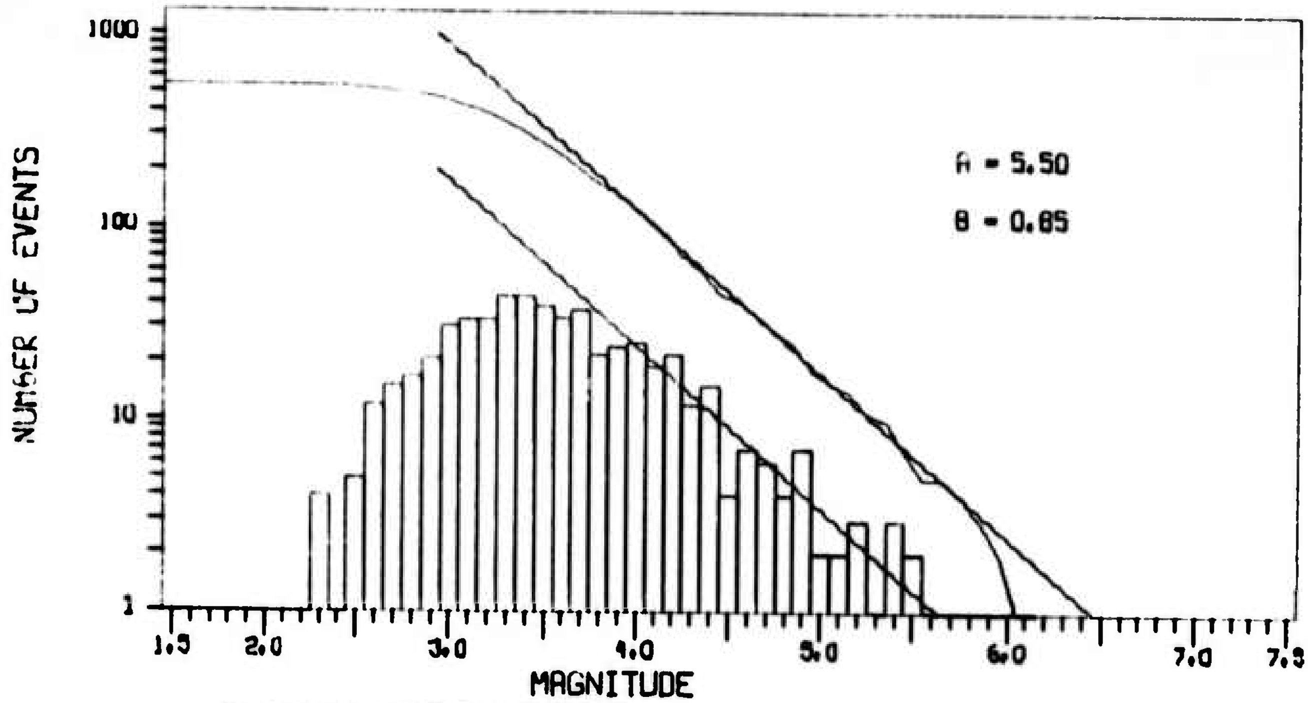
EVENT MAGNITUDE DISTRIBUTION
VLPE CONVENTIONAL MS



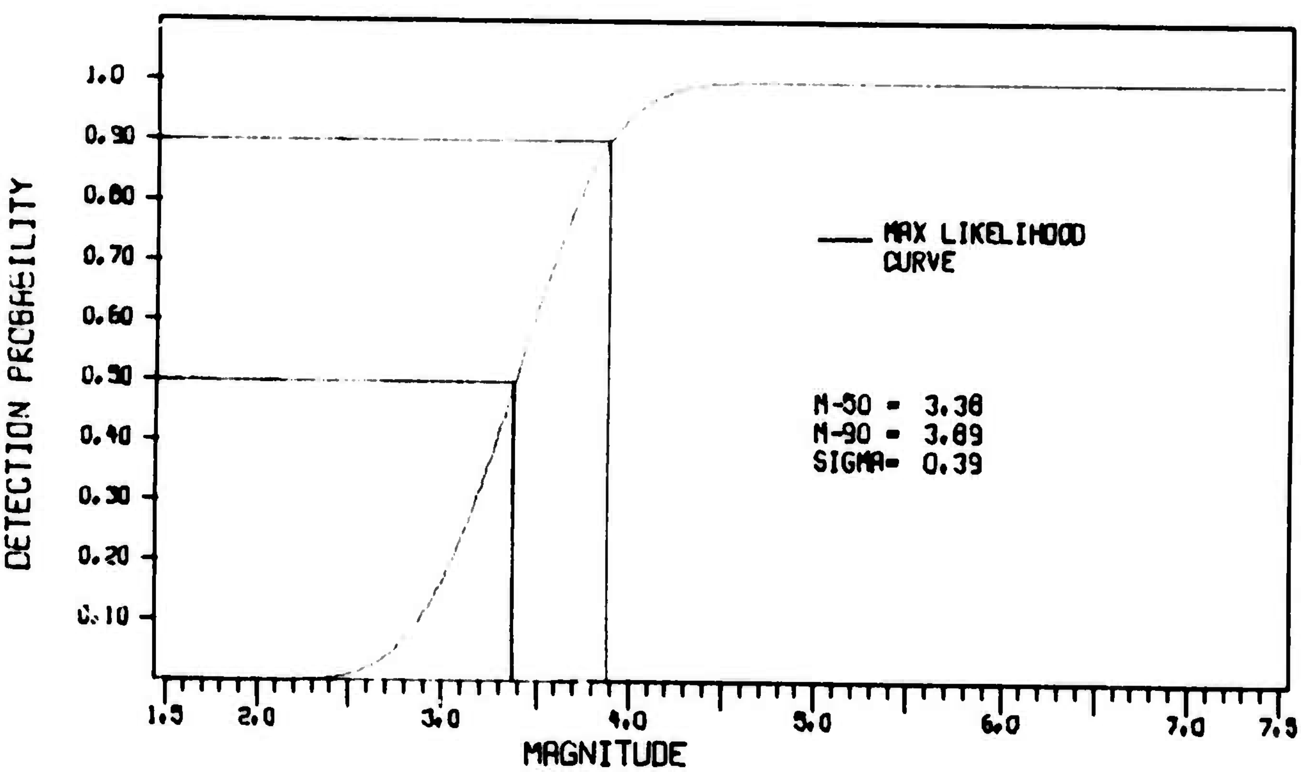
MAXIMUM LIKELIHOOD DETECTABILITY CURVE
VLPE CONVENTIONAL MS

FIGURE IV-3

ESTIMATES OF SEISMICITY AND NETWORK DETECTION PROBABILITY (AT LEAST ONE STATION DETECTING) FOR THE VLPE NETWORK BASED UPON CONVENTIONAL M_s ESTIMATES



EVENT MAGNITUDE DISTRIBUTION
VLPE MS MAX LIKELIHOOD



MAXIMUM LIKELIHOOD DETECTABILITY CURVE
VLPE MS MAX LIKELIHOOD

FIGURE IV-4

ESTIMATES OF SEISMICITY AND NETWORK DETECTION PROBABILITY (AT LEAST ONE STATION DETECTING) FOR THE VLPE NETWORK BASED UPON MAXIMUM LIKELIHOOD M_s ESTIMATES

certain value M. The incremental network detection curve is assumed to be a cumulative Gaussian distribution function of the form:

$$P(\text{detect } M) = \int_{-\infty}^{M} (2\pi s^2)^{-1/2} \cdot e^{-\frac{(M-M_o)^2}{2s^2}} \qquad \text{(IV-3)}$$

where M_o is the 50 percent network detection threshold and s is a standard deviation.

In this way, Figures IV-3 and IV-4 are interpreted as both representing the estimated seismicity and the VLPE network detection capability based on the same set of data. However, in Figure IV-3 conventional network magnitude calculations have been used, while Figure IV-4 refers to maximum likelihood magnitudes. Two important differences are observed between the two cases:

- The maximum likelihood approach yields a small, but significant, decrease in the estimated value of B (0.85 versus 0.93)
- The detection thresholds are lower (by 0.2 to 0.3 M_s units) when maximum likelihood magnitudes are used.

Both of these differences were expected from theoretical considerations. The important questions, of course, is which of the two methods provides the most reliable estimates.

For the detection thresholds, we may compare the results from Figures IV-3 and IV-4 to those obtained independently by Lambert et al. (1974). The 50 percent estimate is the most reliable (Ringdal, 1974) and will be used. Lambert et al. found a 50 percent incremental detection threshold for VLPE of 4.2 m_b units by using the WWSSN combined with LASA and NORSAR as a

reference. This corresponds to about 3.4 M_s units, using their empirically derived conversion formula:

$$M_s = m_b - 0.80 + r \qquad (4.2 \leq m_b \leq 5.5) \qquad \text{(IV-4)}$$

where r is a random Gaussian variable of zero mean. Using NORSAR and ALPA M_s values as a reference, and correcting for transmission path effects, they obtained another, independent estimate of 3.2 for the 50 percent threshold. These values are in good correspondence with the value of 3.38 obtained by the maximum likelihood M_s estimation, while the value of 3.64 yielded by the conventional technique appears to be too high.

With respect to the difference in B-value, it is not easy to find a 'correct' estimate to compare against. Most recent studies have concentrated on b-values for the corresponding magnitude-frequency relationship for bodywaves. For example, Bungum and Husebye (1974), using NORSAR m_b data, estimated regional b-values. For the three regions where most of our events were concentrated (central Asia, southern-eastern Asia and Japan-Kuriles-Kamchatka) their estimates were 0.85, 0.86, and 0.84, respectively. This is in surprisingly good (and probably somewhat coincindental) correspondence with the maximum likelihood result of 0.85 (Figure IV-4). It is significantly lower than the slope of 0.93 in Figure IV-3, estimated by the conventional method.

Some precautions are clearly necessary here. First of all, one might ask whether the NORSAR values represent the 'true' slopes, since they are based upon measurements at only one station. However, it was shown by Ringdal (1974, Appendix A), that under the assumption of a Gaussian magnitude distribution model, with constant variance, the estimate of the slope b by one station will be unbiased relative to the true value. In fact, for precisely the reasons discussed in this report, single station seismicity

estimates may be more reliable than estimates based upon network magnitudes.

Secondly, it is not clear that $B = b$; i.e., that the slopes based on M_s and m_b are identical. However, in the magnitude range where the relationship (IV-4) is valid, this can be shown to be the case.

In conclusion, we feel that it has been demonstrated that the maximum likelihood network magnitude estimation method gives more reliable estimates than the conventional averaging technique, and, if used properly, can provide significant improvements in statistical analysis concerning event detection, discrimination, and seismicity.

SECTION V
CONCLUSIONS AND RECOMMENDATIONS

The maximum likelihood technique has been found to potentially yield a significant improvement in network magnitude estimates of small and medium size events compared to the conventional method of averaging the magnitudes of all detecting stations. Such an improvement has been actually observed both for simulated networks, for a 13 station subnet of the WWSSN, and for the 11 station VLPE network. From our investigations it appears that the maximum likelihood method applied to a 10 station network will provide essentially unbiased magnitude estimates at about 1 magnitude unit lower than the conventional technique does.

We recommend that further research be carried out in order to obtain more complete data on the maximum likelihood method and its underlying assumptions. Specifically, the following topics are suggested:

- Further verification of the Gaussian model for world-wide seismic magnitude distribution for a given event; especially for surface wave magnitudes

- More precise determination of the standard deviation σ in the above distribution, and its possible variation with source function, event depth, magnitude, and seismic region

- Comparison of σ for array station networks and σ for single station networks

- Actual application of the maximum likelihood technique to existing and planned networks. For such applications, it is recommended that

- All threshold values of non-detecting stations should be actually measured for each event detected by the network
- A narrow, but realistic range of σ should be specified, and the likelihood function (II-5) should be maximized as a function of two parameters (μ, σ).

It is strongly recommended that in any future operational network all threshold magnitudes for non-detecting stations should be measured along with the magnitudes of the detecting stations. It is felt that realistic estimates of the magnitudes of small and intermediate events will ultimately have to take all of this information into account, whether or not the techniques and models used in this report are to be applied.

SECTION VI
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APPENDIX A

DERIVATION OF CRAMER-RAO BOUNDS FOR THE MAXIMUM LIKELIHOOD ESTIMATOR

This appendix presents a detailed, formal derivation of the Cramer-Rao bounds on the variance of the maximum likelihood estimator developed in this report. Since it was established in Section II that the estimator is asymptotically normally distributed with minimum variance, these bounds may be used as approximations to the actual variance in practical cases, assuming that the number of stations in the network is sufficiently large.

We use in this appendix a slightly more general problem formulation than in Section II, in that we allow for individual station differences in magnitude bias and signal variance. The statistical test situation is thus summarized as follows:

- Seismic records are examined for a total of n stations
- For station i $(i = 1, 2, \ldots, n)$, it is assumed that the signal magnitude M_i is normally distributed $M_i \sim N(\mu + b_i, \sigma_i^2)$.
- Each station has a certain detection threshold a_i measured at the time of the event; i.e., the station detects the event if $m_i \geq a_i$ and does not detect if $m_i < a_i$. Here, m_i denotes the actual value of the random variable M_i, and is defined regardless of whether or not it actually can be measured
- M_i and M_j are independent random variables for all pairs (i, j); $i \neq j$.

In the following, we assume that the values (b_i, σ_i), $i = 1,2, \ldots, n$ are known a priori, and that

- Threshold magnitudes $a_1, \ldots, a_n$ are actually measured for all stations at the time of expected signal arrival
- Actual event magnitudes $\{m_i ; i \epsilon \mathscr{D}\}$ are measured for all the stations that detect. Here, $\mathscr{D}$ denotes the set of indices corresponding to the detecting stations.

We thus assume:

$$m_i \geq a_i \qquad \text{for all } i \epsilon \mathscr{D}$$

$$m_i < a_i \qquad \text{for all } i \notin \mathscr{D} \qquad \text{(A-1)}$$

The likelihood function L, i.e., the probability of a certain occurrence of detections, no detections, and measured values m_j can now be derived:

$$L(m_1, \ldots, m_n / \mu) \cdot dm_1 \cdots dm_k \cdots =$$

$$\text{Prob}\Big(\bigcap_{i \epsilon \mathscr{D}} M_i \epsilon < m_i, m_i + dm_i > \cap \bigcap_{j \notin \mathscr{D}} M_j < a_j \Big) =$$

$$\prod_{i \epsilon \mathscr{D}} \phi_i(m_i) \cdot dm_i \cdot \prod_{j \notin \mathscr{D}} \Phi_j(a_j) \qquad \text{(A-2)}$$

assuming the boundary conditions (A-1) are satisfied.

Hence:

$$L(m_1, \ldots, m_n / \mu) = \begin{cases} \prod_{i \epsilon \mathscr{D}} \phi_i(m_i) \cdot \prod_{j \notin \mathscr{D}} \Phi_j(a_j) & \text{if (A-1) is satisfied} \\ 0 & \text{elsewhere} \end{cases} \qquad \text{(A-3)}$$

where

$$\phi_i(m_i) = \frac{1}{\sqrt{2\pi}\cdot\sigma_i}\cdot e^{-\frac{(m_i-(\mu+b_i))^2}{2\sigma_i^2}} \qquad \text{(A-4)}$$

$$\Phi_i(a_i) = \int_{-\infty}^{a_i} \phi_i(m_i)\, dm_i \qquad i = 1,2,\ldots n\,. \qquad \text{(A-5)}$$

Some remarks about the likelihood function are appropriate at this point:

- Although all magnitudes $m_1, \ldots m_n$ and threshold values $a_1 \ldots a_n$ are included in the boundary conditions (A-1) of the likelihood function L, only the subsets $\{m_i,\ i \in \mathcal{D}\}$ and $\{a_j,\ j \notin \mathcal{D}\}$ actually enter the expression of the value of L (A-3). Therefore, L, as a function of μ, may be maximized as long as the magnitudes of the detecting stations and the threshold values of the non-detecting stations are known
- It is clear that (A-3) also applies in the two special cases when either all stations detect or no stations detect. Note that in the latter case, the likelihood function will be monotonically decreasing, and hence cannot be maximized.

We now proceed to compute the Cramer-Rao bounds on the variance of an unbiased estimator $\hat{\mu}$ of the parameter μ. From Cramer (1945), the minimum variance of $\hat{\mu}$ is given by

$$\left[\mathrm{Var}\,(\hat{\mu})\right]^{-1} = E\left(\frac{\partial \log L}{\partial \mu}\right)^2 \qquad \text{(A-6)}$$

where E denotes the statistical expectation, with the sample space consisting of all possible combinations of station magnitudes $m_1, \ldots m_n$. Of course,

with all threshold magnitudes $a_1, \ldots, a_n$ known a priori, a specification of a particular set of $m_1, \ldots, m_n$ will automatically imply knowledge of which stations count as detections.

In order to evaluate (A-6), we find it convenient to introduce a set of functions L_i given by

$$L_i(m_i/\mu) = \begin{cases} \phi_i(m_i) & \text{if } m_i \geq a_i \\ \Phi_i(a_i) & \text{if } m_i < a_i \end{cases} . \tag{A-7}$$

It is clear from the definition (A-3) and the boundary conditions (A-1) that we may write

$$L(m_1, \ldots, m_n/\mu) = \prod_{i=1}^{n} L_i(m_i/\mu) . \tag{A-8}$$

Furthermore, since L_i and L_j for $i \neq j$ relate to two independent observations, (A-8) may be interpreted as a product of independent functions of random variables (strictly speaking; with m_i replaced with M_i). The log likelihood function becomes:

$$\log L(m_1, \ldots m_n/\mu) = \sum_{i=1}^{n} \log L_i(m_i/\mu) . \tag{A-9}$$

Hence

$$E\left(\frac{\partial \log L}{\partial \mu}\right)^2 = E\left(\sum_{i=1}^{n} \frac{\partial \log L_i}{\partial \mu}\right)^2 \tag{A-10}$$

Now, it is true in general that for a set $\{x_1, \ldots, x_n\}$ of independent random variables, we have

$$E\left(\sum_{i=1}^{n} X_i\right)^2 = \sum_{i=1}^{n} EX_i^2 + \sum_{i \neq j} EX_i \quad EX_j \ . \qquad \text{(A-11)}$$

Therefore:

$$E\left(\frac{\partial \log L}{\partial \mu}\right)^2 = \sum_{i=1}^{n} E\left(\frac{\partial \log L_i}{\partial \mu}\right)^2 + \sum_{i \neq j} E\left(\frac{\partial \log L_i}{\partial \mu}\right) \cdot E\left(\frac{\partial \log L_j}{\partial \mu}\right) \qquad \text{(A-12)}$$

We now evaluate the relevant terms, and first observe that

$$\frac{\partial \log L_i}{\partial \mu} = \begin{cases} \dfrac{1}{\phi_i(m_i)} \cdot \dfrac{\partial}{\partial \mu} \phi_i(m_i) & \text{for } m_i \geq a_i \\ \\ \dfrac{1}{\Phi_i(a_i)} \cdot \dfrac{\partial}{\partial \mu} \Phi_i(a_i) & \text{for } m_i < a_i \end{cases} \qquad \text{(A-13)}$$

Thus:

$$E\left(\frac{\partial \log L_i}{\partial \mu}\right) = \int_{-\infty}^{\infty} \frac{\partial \log L_i}{\partial \mu} \cdot \phi_i(m_i) \, dm_i =$$

$$= \int_{a_i}^{\infty} \frac{\partial}{\partial \mu} \phi_i(m_i) dm_i + \Phi_i(a_i)^{-1} \cdot \frac{\partial}{\partial \mu} \Phi_i(a_i) \cdot \int_{-\infty}^{a_i} \phi_i(m_i) \, dm_i$$

$$= \frac{\partial}{\partial \mu} (1 - \Phi_i(a_i)) + \frac{\partial}{\partial \mu} \Phi_i(a_i) = 0 \qquad \text{(A-14)}$$

where we have used the definition (A-5) of Φ_i and reversed the order of derivation and integration. It is interesting to note that (A-14) is valid regardless of the actual type of probability density function ϕ_i.

The remaining terms of equation (A-12) are found by evaluating:

$$E\left(\frac{\partial \log L_i}{\partial \mu}\right)^2 = \int_{-\infty}^{\infty} \left(\frac{\partial \log L_i}{\partial \mu}\right)^2 \cdot \phi_i(m_i)\, dm_i \quad . \qquad \text{(A-15)}$$

By inserting the actual normal probability density function defined by (A-4) in (A-13) we obtain:

$$\frac{\partial \log L_i}{\partial \mu} = \begin{cases} \dfrac{1}{\sigma_i^2} \cdot \left(\dfrac{m_i - (\mu + b_i)}{\sigma_i}\right)^2 & \text{for } m_i \geq a_i \\[2ex] \dfrac{1}{\Phi_i(a_i)} \cdot \dfrac{1}{\sigma_i^2} \cdot \phi_i(a_i) & \text{for } m_i < a_i \quad . \end{cases} \qquad \text{(A-16)}$$

When these expressions are substituted in (A-15) and the computations carried out, we arrive at the following result:

$$E\left(\frac{\partial \log L_i}{\partial \mu}\right)^2 = \frac{1}{\sigma_i^2} \cdot \left[\frac{a_i - (\mu + b_i)}{\sigma_i^2} \cdot \phi_i(a_i) + (1 - \Phi_i(a_i)) + \frac{\left[\phi_i(a_i)\right]^2}{\sigma_i^2\, \Phi_i(a_i)}\right] \qquad \text{(A-17)}$$

We can relate this expression to the standard Gaussian distribution by substituting

$$Z_i = \frac{a_i - (\mu + b_i)}{\sigma_i} \qquad \text{(A-18)}$$

Consequently:

$$\phi_i(a_i) = \frac{1}{\sigma_i} \cdot \phi(Z_i) \qquad \text{(A-19)}$$

$$\Phi_i(a_i) = \Phi(Z_i) \tag{A-20}$$

where Φ and ϕ represent the standard (0, 1) Gaussian distribution and density functions, respectively. With this notation, (A-17) becomes:

$$\boxed{E\left(\frac{\partial \log L_i}{\partial \mu}\right)^2 = \frac{1}{\sigma_i^2}\left[Z_i \cdot \phi(Z_i) + (1 - \Phi(Z_i)) + \frac{\left[\phi(Z_i)\right]^2}{\Phi(Z_i)}\right]} \tag{A-21}$$

Further, by recalling (A-6) and (A-12), and observing that, by (A-14), the cross terms in (A-12) vanish, we have the result:

$$\boxed{\mathrm{Var}(\hat{\mu}) = 1 \Big/ \sum_{i=1}^{n} E\left(\frac{\partial \log L_i}{\partial \mu}\right)^2} \tag{A-22}$$

where the individual terms may be computed using (A-21).

The following comments to this result are appropriate: First, let $a_i \rightarrow -\infty$, $i = 1, 2, \ldots, n$; this is equivalent to saying that all stations detect with probability 1. In this particular case, we obtain from (A-22)

$$\mathrm{Var}\ \hat{\mu} = \left[\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right]^{-1}. \tag{A-23}$$

Note that the likelihood function (A-3) is simplified in this case by allowing no non-detections, and the maximum likelihood estimator can be computed analytically:

$$\hat{\mu} = \sum_{i=1}^{n}\left[\frac{m_i - b_i}{\sigma_i^2}\right] \cdot \left[\sum_{i=1}^{n} \frac{1}{\sigma_i^2}\right]^{-1}. \tag{A-24}$$

This is indeed the minimum variance unbiased estimator in this special case, and its variance is given by (A-23). A more familiar expression is found by setting all $b_i = 0$, $i = 1, 2 \cdots n$ and assuming all $\sigma_i = \sigma$ $i = 1, 2 \cdots n$. We then obtain the standard expression

$$\hat{\mu} = \frac{1}{n} \cdot \sum_{i=1}^{n} m_i \tag{A-25}$$

$$\text{Var } \hat{\mu} = \frac{1}{n} \cdot \sigma^2 \tag{A-26}$$

for estimating the mean μ of a normal distribution.

However, it is important to notice that in a practical situation (when $a_i \neq -\infty$; $i = 1, 2, \ldots n$), the detection threshold of the stations will affect the variance of the estimate of μ, even for a station that detects a given event. This is not unreasonable, since the variance reflects what would happen if (hypothetically) a given random experiment was repeated a large number of times. Depending upon its detection threshold, a station would then be expected to have a certain percentage of non-detections, and this is reflected in the expression for the Cramer-Rao bound.

A quantitative estimate of the contribution of a given station to the reduction in variance of the estimate of μ can be obtained by inspecting Table A-1. It is seen that the weighting factor in (A-21) is close to 1 unless the station's noise level is significantly higher than the actual estimated event magnitude.

TABLE A-1

TABLE SHOWING VALUES OF THE WEIGHTING FACTORS ENTERING INTO THE CRAMER-RAO BOUNDS ON THE VARIANCE OF THE MAXIMUM LIKELIHOOD ESTIMATOR

Z	W(Z)
$-\infty$	1.00
-1.0	0.97
-0.5	0.92
0.0	0.82
0.5	0.66
1.0	0.47
1.5	0.28
2.0	0.13
2.5	0.05
3.0	0.015
$+\infty$	0.00

$$W(Z) = Z \cdot \phi(Z) + (1 - \Phi(Z)) + \frac{[\phi(Z)]^2}{\Phi(Z)}$$

where $Z = \dfrac{a - (b + \mu)}{\sigma}$

Φ and ϕ are the standard Gaussian distribution and density functions, respectively.